On the GGS Conjecture

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Abstract

In the 1980's, Belavin and Drinfeld classified solutions r of the classical Yang-Baxter equation (CYBE) for simple Lie algebras $\mathfrak{g}$ satisfying $0 \neq r + r_{21} \in (S^2\mathfrak{g})^{\mathfrak{g}}$ [1]. They proved that all such solutions fall into finitely many continuous families and introduced combinatorial objects to label these families, Belavin-Drinfeld triples. In 1993, Gerstenhaber, Giaquinto, and Schack attempted to quantize such solutions for Lie algebras $\mathfrak{sl}(n)$. As a result, they formulated a conjecture stating that certain explicitly given elements $R \in Mat_n(\mathbb{C}) \otimes Mat_n(\mathbb{C})$ satisfy the quantum Yang-Baxter equation (QYBE) and the Hecke relation [5]. Specifically, the conjecture assigns a family of such elements R to any Belavin-Drinfeld triple of type A_{n-1}. Following a suggestion from Gerstenhaber and Giaquinto, we propose an alternate form for R, given by $R_J = q^{r^0} J^{-1} R_s J_{21} q^{r^0}$, for a suitable twist J and a diagonal matrix r^0, where R_s is the standard Drinfeld-Jimbo solution of the QYBE. We formulate the "twist conjecture", which states that $R_J = R_{\mathrm{GGS}}$ and that R_J satisfies the QYBE. Since R_J by construction satisfies the Hecke relation, this conjecture implies the GGS conjecture. We check the twist conjecture by computer for $n \leq 12$ and show that it is true modulo $\hbar^3$. We provide combinatorial formulas for coefficients in the matrices R_J, R_{GGS} and prove both conjectures in the *orthogonal generalized disjoint case*—where $\Gamma_1 = \bigcup_i \Gamma_1^i$ with $\Gamma_1^i \perp \Gamma_1^j, i \neq j$, $\tau\Gamma_1^i \cap \Gamma_1 \subset \Gamma_1^{i+1}$, and $\tau^j \Gamma_1^i \perp \Gamma_1^i, \forall i, j \geq 1$. We also prove the twist conjecture in the *disjoint case*, $\Gamma_1 \cap \Gamma_2 = \emptyset$. Finally, we prove the twist conjecture for the Cremmer-Gervais triple and discuss cases in which it is known that $R_J = R_{\mathrm{GGS}}$.

1 Main Results

We begin this section by introducing Belavin-Drinfeld triples. We present the GGS conjecture, which is motivated by calculating possible quantizations of r modulo $\hbar^3$. Next, we proceed to formulate the twist conjecture and give the remarkably similar combinatorial descriptions of the twist and the GGS R-matrix. Finally, we summarize our main results, namely the computer verification of the twist conjecture, its proof modulo $\hbar^3$, and a complete proof of the twist conjecture in the disjoint, orthogonal generalized disjoint, and Cremmer-Gervais cases.

1.1 Belavin-Drinfeld triples

Let $(e_i), 1 \leq i \leq n$, be a basis for $\mathbb{C}^n$. Set $\Gamma = \{e_i - e_{i+1} : 1 \leq i \leq n-1\}$. We will use the notation $\alpha_i \equiv e_i - e_{i+1}$. Let $(,)$ denote the inner product on $\mathbb{C}^n$ having (e_i) as an orthonormal basis.

Definition 1.1 *[1] A Belavin-Drinfeld triple of type A_{n-1} is a triple $(\tau, \Gamma_1, \Gamma_2)$ where $\Gamma_1, \Gamma_2 \subset \Gamma$ and $\tau : \Gamma_1 \to \Gamma_2$ is a bijection, satisfying two conditions:*

(a) $\forall \alpha, \beta \in \Gamma_1$, $(\tau\alpha, \tau\beta) = (\alpha, \beta)$.

(b) τ is nilpotent: $\forall \alpha \in \Gamma_1, \exists k \in \mathbb{N}$ such that $\tau^k \alpha \notin \Gamma_1$.

Let $\mathfrak{g} = \mathfrak{gl}(n)$ be the Lie algebra of $n \times n$ matrices. (Although $\mathfrak{gl}(n)$ is not simple, solutions correspond to those in $\mathfrak{sl}(n)$, and it will simplify computations. For the same reason, we state the GGS and twist conjectures in $\mathfrak{gl}(n)$.) Set $\mathfrak{h} \subset \mathfrak{g}$ to be the subset of diagonal matrices. Elements of $\mathbb{C}^n$ define linear functions on $\mathfrak{h}$ by $\big(\sum_i \lambda_i e_i\big)\big(\sum_i a_i\, e_{ii}\big) = \sum_i \lambda_i a_i$. Let $P = \sum_{1 \leq i,j \leq n} e_{ij} \otimes e_{ji}$ be the Casimir element for $\mathfrak{g}$ as well as the permutation matrix, and let $P^0 = \sum_i e_{ii} \otimes e_{ii}$ be the projection of P to $\mathfrak{h} \otimes \mathfrak{h}$.

For any Belavin-Drinfeld triple, consider the following equation for $r^0 \in \mathfrak{h} \wedge \mathfrak{h}$:

$$\forall \alpha \in \Gamma_1,\, \big[(\alpha - \tau\alpha) \otimes 1\big] r^0 = \frac{1}{2}\big[(\alpha + \tau\alpha) \otimes 1\big] P^0. \tag{1.1}$$

Belavin and Drinfeld showed that solutions $r \in \mathfrak{g} \otimes \mathfrak{g}$ of the CYBE satisfying $r + r^{21} = P$, up to isomorphism, are given by a discrete datum (the Belavin-Drinfeld triple) and a continuous datum (a solution $r^0 \in \mathfrak{h} \wedge \mathfrak{h}$ of (1.1)). We now describe this classification. For $\alpha = e_i - e_j$, set $e_\alpha \equiv e_{ij}$, and say $\alpha > 0$ if $i < j$, and otherwise $\alpha < 0$. Define $|\alpha| = |j - i|$. For any $Y \subset \Gamma$, set $\tilde{Y} = \{v \in \mathrm{Span}(Y) \mid v = e_i - e_j, v > 0\}$; in particular we will use $\tilde{\Gamma}_1, \tilde{\Gamma}_2$. We extend τ additively to a map $\tilde{\Gamma}_1 \to \tilde{\Gamma}_2$, i.e. $\tau(\alpha + \beta) = \tau\alpha + \tau\beta$. Whenever $\tau^k\alpha = \beta$ for $k \geq 1$, we say $\alpha \prec \beta$. Clearly $\prec$ is a partial ordering on $\tilde{\Gamma}$. Finally, for any $\beta = \tau^k\alpha$, $\alpha = e_j - e_i, \beta = e_k - e_{k+i-j}, j < i - 1$, we say τ^k reverses orientation on α if $\tau\alpha_j = \alpha_{k+i-j-1}$ and τ^k preserves orientation on α if $\tau\alpha_j = \alpha_k$. In the reversing case, write $\alpha \prec^{\leftarrow} \beta$ and $\mathrm{sign}(\alpha, \beta) = (-1)^{1-|\alpha|}$; in the preserving case write $\alpha \prec^{\rightarrow} \beta, \mathrm{sign}(\alpha, \beta) = 1$. We set $\mathrm{sign}(\alpha, \beta) = 1$ when $j + 1 = i$. Set $x \wedge y \equiv x \otimes y - y \otimes x$ for $x, y \in Mat_n(\mathbb{C})$ and $z = \sum_{i,j,k,l} z_{ik}^{jl} e_{ij} \otimes e_{kl}$ for $z \in Mat_n(\mathbb{C}) \otimes Mat_n(\mathbb{C})$. Then we define

$$a = \sum_{\alpha \prec \beta} \mathrm{sign}(\alpha, \beta)\, e_{-\alpha} \wedge e_\beta, \quad r_s = \frac{1}{2}\sum_i e_{ii} \otimes e_{ii} + \sum_{\alpha > 0} e_{-\alpha} \otimes e_\alpha, \quad r = r^0 + a + r_s, \tag{1.2}$$

where $r_s \in \mathfrak{g} \otimes \mathfrak{g}$ is the standard solution of the CYBE satisfying $r_s + r_s^{21} = P$, and r is the solution corresponding to the data $((\Gamma_1, \Gamma_2, \tau), r^0)$. It follows from [1] that any solution $\tilde{r} \in \mathfrak{g}, \tilde{r} + \tilde{r}_{21} = P$ is equivalent to such a solution r under an automorphism of $\mathfrak{g}$.

1.2 The GGS conjecture

The GGS conjecture proposes a hypothetical quantization of the matrix r given in (1.2), given by a matrix $R \in Mat_n(\mathbb{C}) \otimes Mat_n(\mathbb{C})$ conjectured to satisfy the quantum Yang-Baxter equation (QYBE), $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$, and the Hecke relation, $(PR - q)(PR + q^{-1}) = 0$. This may be formulated and justified as follows (which is more or less the original motivation).

If we write $R \equiv 1 + 2\hbar r + 4\hbar^2 s \pmod{\hbar^3}$, where $q \equiv e^\hbar$, then we can consider the constraints imposed by the QYBE and the Hecke relation modulo $\hbar^3$. One may easily check that QYBE becomes the CYBE for r, while the Hecke relation becomes the condition $s + s_{21} = r^2$. Thus, there is a unique choice of s that is symmetric, namely $\frac{1}{2}r^2 = \frac{1}{2}((r^0)^2 + ar^0 + r^0 a + \epsilon)$ where

$$\epsilon = ar_s + r_s a + a^2. \tag{1.3}$$

Proposition 1.1 *There exist unique polynomials $P_{i,j,k,l}$ of the form $xq^y(q-q^{-1})^z, x, y \in \mathbb{C}, z \in \{0,1\}$ such that $\sum_{i,j,k,l} P_{i,j,k,l} e_{ij} \otimes e_{kl} \equiv 1 + 2\hbar r + 2\hbar^2 r^2 \pmod{\hbar^3}$.*

Proof. The proof is easy. $\square$

Definition 1.2 *Define $R_{GGS} = \sum_{i,j,k,l} P_{i,j,k,l} e_{ij} \otimes e_{kl}$, with the $P_{i,j,k,l}$ uniquely determined by Proposition 1.1. The matrix R_{GGS} is called the GGS R-matrix.*

Define the following matrices:

$$\tilde{a} = \sum_{i,j,k,l} a_{ik}^{jl} q^{a_{ik}^{jl} \epsilon_{ik}^{jl}} e_{ij} \otimes e_{kl}, \quad \bar{R}_{\mathrm{GGS}} = R_s + (q - q^{-1})\tilde{a}, \tag{1.4}$$

where $R_s = q \sum_i e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i>j} e_{ij} \otimes e_{ji}$ is the standard Drinfeld-Jimbo solution to the QYBE.

Proposition 1.2 *The matrix $R_{GGS} = q^{r^0} \bar{R}_{GGS} q^{r^0}$.*

Proof. This is clear. $\square$

Remark 1.1 We see that $R_{\mathrm{GGS}} \equiv q^{2r} \pmod{\hbar^3}$, although $R_{\mathrm{GGS}} \neq q^{2r}$ in general.

Conjecture 1.1 "the GGS conjecture" [5]
I. The matrix R_{GGS} satisfies the QYBE.
II. The matrix R_{GGS} satisfies the Hecke relation.

We will sometimes refer separately to the two parts as Conjectures 1.1.I and 1.1.II.

Remark 1.2 It is sufficient to check the QYBE for one r^0 since the space of homogeneous solutions to (1.1) is exactly the space $\Lambda^2(\mathfrak{l})$ where $\mathfrak{l} \subset \mathfrak{h}$ is the space of symmetries of the Belavin-Drinfeld triple, i.e. $(x, \alpha) = (x, \tau\alpha)$ for any $x \in \mathfrak{l}, \alpha \in \Gamma_1$. It is easy to see that $x \in \mathfrak{l}$ implies $[1 \otimes x + x \otimes 1, R_{\mathrm{GGS}}] = 0$, and it follows for any $y \in \Lambda^2(\mathfrak{l})$ that $q^y R_{\mathrm{GGS}} q^y$ satisfies the QYBE iff R_{GGS} does. The same holds for R_J as defined in the following section.

Now, we describe our new results on the GGS conjecture.

Theorem 1.1 *(i) The GGS conjecture is true for $n \leq 12$. (ii) The GGS conjecture is true modulo $\hbar^3$.*

Proof. (i) This has been verified by the author through computer programs detailed in [10]. The programs check the QYBE and Hecke relation directly using one choice of r^0. One may check that this is sufficient to prove GGS for any r^0.

(ii) This is obvious from construction. $\square$

We see that the strangest matrix in the definition of R_{GGS} is ϵ. Here we give a simple combinatorial formula for this unusual matrix. For $i < j, k < l$, we say that $e_i - e_j \lessdot e_k - e_l$ iff $j = k$, and similarly define $\gtrdot$. Let [statement] $= 1$ if "statement" is true and 0 if "statement" is false.

Proposition 1.3 *We may rewrite ϵ as follows:*

$$\epsilon = \sum_{\alpha \prec \beta} sign(\alpha, \beta)\big[-\frac{1}{2}[\alpha \lessdot \beta] - \frac{1}{2}[\beta \lessdot \alpha] - [\exists \gamma, \alpha \prec \gamma \prec \beta, \alpha \lessdot \gamma]$$
$$- [\exists \gamma, \alpha \prec \gamma \prec \beta, \alpha \gtrdot \gamma] + [\alpha \prec^{\leftarrow} \beta](1 - |\alpha|)\big](e_\beta \otimes e_{-\alpha} + e_{-\alpha} \otimes e_\beta) \quad (1.5)$$

Proof. Given in Section 2. □

Example 1.1 For a given n, there are exactly $\phi(n)$ triples (ϕ is the Euler ϕ-function) in which $|\Gamma_1| + 1 = |\Gamma|$ [4]. These are are called *generalized Cremmer-Gervais* triples. These are indexed by $m \in \mathbb{Z}^+$, where $\gcd(n, m) = 1$, and given by $\Gamma_1 = \Gamma \setminus \{\alpha_{n-m}\}$, $\Gamma_2 = \Gamma \setminus \{\alpha_m\}$, and $\tau(\alpha_i) = \alpha_{\mathrm{Res}(i+m)}$, where Res gives the residue modulo n in $\{1, \ldots, n\}$. For these triples, there is a unique r^0 with first component having trace 0, which is given by $(r^0)^{ii}_{ii} = 0, \forall i$, and $(r^0)^{ij}_{ij} = \frac{1}{2} - \frac{1}{n}\mathrm{Res}(\frac{j-1}{m})$ (this is easy to verify directly and is also given in [4]). With this r^0, R_{GGS} has a very nice combinatorial formula, which was conjectured by Giaquinto and checked in some cases. We now state and prove this formula. As in [6], define $e_{-\alpha} \wedge_c e_\beta = q^{-c} e_{-\alpha} \otimes e_\beta - q^c e_\beta \otimes e_{-\alpha}$. Let $O(\alpha, \beta) = l$ when $\tau^l \alpha = \beta$.

Proposition 1.4 *R_{GGS} is given as follows:*

$$R_{GGS} = q^{r^0} R_s q^{r^0} + \sum_{\alpha \prec \beta} (q - q^{-1}) e_{-\alpha} \wedge_{\frac{-2O(\alpha,\beta)}{n}} e_\beta. \quad (1.6)$$

Proof. See Appendix B. □

Remark 1.3 Our formulation is from [6], correcting misprints. The original formulation in [5] is somewhat different. We will write $x_{q^{-1}}$ to denote the matrix x with q^{-1} substituted for q. Define $(x \otimes y)^T = x^T \otimes y^T$ where x^T is the transpose of x, for $x, y \in Mat_n(\mathbb{C})$. Then, the original form of R_{GGS} can be written as follows:

$$R = q^{-r^0}\big(R_s + (q^{-1} - q)\tilde{a}^T_{q^{-1}}\big)q^{-r^0}.$$

Denoting R as this matrix and R_{GGS} as given before, we have $R_{\mathrm{GGS}} - R^T_{q^{-1}} = q^{r^0}(q - q^{-1})P q^{r^0} = (q - q^{-1})P$. Thus, R_{GGS} satisfies the Hecke relation iff R satisfies the Hecke relation. In this case, we have $PR^T_{q^{-1}} = (PR_{\mathrm{GGS}})^{-1}$, so $R^T_{q^{-1}} = (R^{-1}_{\mathrm{GGS}})_{21}$, and thus R satisfies the QYBE iff R_{GGS} does. Thus, the two formulations are equivalent.

1.3 The twist conjecture

In [2], it is proved that any quasitriangular structure as defined in Section 1.1 has a quantization which is a twist of the standard quasitrangular Hopf algebra $U_q(\mathfrak{gl}(n))$. In [7] (see also [3]), such a twist is constructed for the *disjoint* case, $\Gamma_1 \cap \Gamma_2 = \emptyset$ (another twist is given in Appendix A.2). Thus, in the n-dimensional representation, there should exist $J \in Mat_n(\mathbb{C}) \otimes Mat_n(\mathbb{C})$ so that $R_J = q^{r^0} J^{-1} R_s J_{21} q^{r^0}$ satisfies the QYBE and Hecke relation. Further, it is especially nice to look for *triangular* twists: twists where $J = 1 + N$ and

$N = \sum_{\alpha,\beta\in\tilde{\Gamma}} N_{\alpha,\beta} e_\beta \otimes e_{-\alpha}$. In this section we give an explicit J of this form designed so that $R_J \equiv R_{\mathrm{GGS}} \pmod{\hbar^3}$ and conjecture that R_J satisfies the QYBE and $R_J = R_{\mathrm{GGS}}$. To find this interesting twist, the author found the Gauss decomposition $\bar{R}_{\mathrm{GGS}} = J^{-1} R_s J_{21}$ (which is necessarily unique, if it exists) for all triples $n \le 12$.

First, we will define some useful notation. Given a matrix

$$x = \sum_{\alpha,\beta>0} N^+(\alpha,\beta) e_\beta \otimes e_{-\alpha} + \sum_{\alpha,\beta>0} N^-(\alpha,\beta) e_{-\alpha} \otimes e_\beta + D,$$

where $D = \sum_i D_i \otimes D_i'$ with D_i, D_i' diagonal, denote

$$x_+ \equiv \sum_{\alpha,\beta>0} N^+(\alpha,\beta) e_\beta \otimes e_{-\alpha} + \frac{1}{2}D, \quad x_- \equiv \sum_{\alpha,\beta>0} N^-(\alpha,\beta) e_{-\alpha} \otimes e_\beta + \frac{1}{2}D,$$

$$x_{\alpha,\beta} = N^+(\alpha,\beta), \quad x_{-\beta,-\alpha} = N^-(\alpha,\beta).$$

Now, we proceed to define J. Set $X = \{(\alpha,\beta) \in \tilde{\Gamma}_1 \times \tilde{\Gamma}_2 \mid \alpha \prec \beta\}$ and $X^i = \{(\alpha,\beta) \in X \mid \tau^i(\alpha) = \beta\}$ so that $X = \cup X^i$. Given any total ordering $<$ on a set Y, we will use $\prod_{x\in Y}^{<}$ to denote a product over all elements of Y, left to right, under the order $<$.

Define the following matrices, products taken left to right, with $K_{\alpha,\beta} \in \mathbb{C}$:

$$A^i = (q - q^{-1}) \sum_{\beta=\tau^i(\alpha)} \mathrm{sign}(\alpha,\beta) q^{K_{\alpha,\beta}} e_\beta \otimes e_{-\alpha} \tag{1.7}$$

$$J^i = 1 + A^i, \quad J = \prod_{i=1}^{d} J^i, \quad \bar{R}_J = J^{-1} R_s J_{21}, \quad R_J = q^{r^0} \bar{R}_J q^{r^0}. \tag{1.8}$$

Proposition 1.5 *There exists an ordering $<$ on X, such that J and J^{-1} are given by the formulas*

$$J = \prod_{(\alpha,\beta)\in X}^{<} \big(1 + \mathit{sign}(\alpha,\beta)(q-q^{-1}) q^{K_{\alpha,\beta}} e_\beta \otimes e_{-\alpha}\big), \tag{1.9}$$

$$J^{-1} = \prod_{(\alpha,\beta)\in X}^{>} \big(1 - \mathit{sign}(\alpha,\beta)(q-q^{-1}) q^{K_{\alpha,\beta}} e_\beta \otimes e_{-\alpha}\big). \tag{1.10}$$

Proof. Indeed, each X^p may be ordered as follows: set $\beta = e_i - e_j, \beta' = e_k - e_l$. Then if $i > k$, $(\alpha,\beta) < (\alpha',\beta')$. If $i = k$ and $j > l$ then $(\alpha,\beta) < (\alpha',\beta')$. Then, it is clear that $\prod^{<}_{(\alpha,\beta)\in X^i} \big(1+\mathrm{sign}(\alpha,\beta)(q-q^{-1})q^{K_{\alpha,\beta}} e_\beta \otimes e_{-\alpha}\big) = J^i$ because, upon expansion, all products $(e_\beta \otimes e_{-\alpha})(e_{\beta'} \otimes e_{-\alpha'})$ vanish. Then, all that is needed is to extend $<$ to an ordering on X given by $X_i < X_j$ whenever $i < j$. $\square$

Remark 1.4 The product formula (1.9) for J is especially natural in light of the formula for the universal R-matrix given in [9]. In Appendix A, the importance of the ordering by powers of τ is demonstrated in the construction of a twist $\mathcal{J} \in U_q(\mathfrak{gl}(n)) \otimes U_q(\mathfrak{gl}(n))$ corresponding to J for the simplest case where τ^2 is defined (i.e. $\Gamma_1 \cap \Gamma_2 \neq \emptyset$). This is the orthogonal generalized disjoint case.

Theorem 1.2 *(i) There exist unique half-integer $K_{\alpha,\beta}$ such that* $\frac{d^2}{d\hbar^2}\Big|_{\hbar=0} \left[\bar{R}_J - (\bar{R}_J)_{21}\right]_{\alpha,\beta} = 0, \forall \alpha \prec \beta$.

(ii) These $K_{\alpha,\beta}$ are given by the combinatorial formula

$$K_{\alpha,\beta} = \frac{1}{2}[\alpha \lessdot \beta, \alpha \prec \beta] - \frac{1}{2}[\alpha \gtrdot \beta, \alpha \prec \beta] + [\exists \gamma \mid \alpha \prec \gamma \prec \beta, \alpha \lessdot \gamma] \\ - [\exists \gamma \mid \alpha \prec \gamma \prec \beta, \alpha \gtrdot \gamma] + [\alpha \prec^{\leftarrow} \beta](1 - |\alpha|). \quad (1.11)$$

(iii) For these $K_{\alpha,\beta}$, and no others, one has $R_J \equiv R_{GGS} \pmod{\hbar^3}$.

Proof. (i) This is clear upon expanding R_J modulo $\hbar^3$. (See Section 2 for details.)
(ii), (iii) Proved in Section 2. □

Conjecture 1.2 "the twist conjecture" *Taking $K_{\alpha,\beta}$ as in* (1.11)*,*
I. The matrix R_J satisfies the QYBE.
II. The matrix R_J coincides with R_{GGS}.

The two parts of the twist conjecture are analogous to those of the GGS conjecture in the following way. Conjecture 1.2.II is a strengthened version of 1.1.II, while Conjecture 1.2.I is equivalent to 1.1.I modulo 1.2.II.

Theorem 1.3 *i) The twist conjecture holds for $n \leq 12$. ii) The twist conjecture is true modulo $\hbar^3$.*

Proof. (i) Given Theorem 1.1, it is sufficient to check $J^{-1}\bar{R}_{\mathrm{GGS}}J_{21} = \bar{R}_J$ for all triples, $n \leq 12$, which has been carried out directly by computer.
(ii) Conjecture 1.2.I mod $\hbar^3$ is obvious from construction, and Conjecture 1.2.II is true mod $\hbar^3$ as a consequence of Theorem 1.2. □

1.4 The generalized disjoint and Cremmer-Gervais triples

Definition 1.3 *A triple $(\Gamma_1, \Gamma_2, \tau)$ is said to be generalized disjoint if $\Gamma_1 = \bigcup_{i=1}^m \Gamma_1^i$ where $\Gamma_1^i \perp \Gamma_1^j, i \neq j$ and $\tau\Gamma_1^i \cap \Gamma_1 \subset \Gamma_1^{i+1}, i < m$, and $\tau\Gamma_1^m \cap \Gamma_1 = \emptyset$. If, in fact, $\tau\Gamma_1^i \perp \Gamma_1^j, j \neq i+1$, and $\tau\Gamma_1^m \perp \Gamma_1$, then the triple is said to be orthogonal generalized disjoint.*

Example 1.2 The case $\Gamma_1 = \{\alpha_i \mid i \not\equiv 0 \pmod 3, i < n-3\}$, $\tau\alpha_i = \alpha_{i+3}$ is orthogonal generalized disjoint.

Theorem 1.4 *(i) The twist conjecture is true in the disjoint case. (ii) The twist conjecture is true in the orthogonal generalized disjoint case.*

Proof. See Sections 3.1, 3.2, and Appendices A.1, A.2. Note that the twist $\mathcal{J}$ used in the disjoint case was first constructed by T. Hodges in [7]. □

Theorem 1.5 *The twist conjecture is true for the Cremmer-Gervais triples* $(\Gamma_1, \Gamma_2, \tau)$ *and* $(\Gamma_2, \Gamma_1, \tau^{-1})$ *where* $\Gamma_1 = \{\alpha_1, \ldots, \alpha_{n-2}\}, \Gamma_2 = \{\alpha_2, \ldots, \alpha_{n-1}\}, \tau\alpha_i = \alpha_{i+1}$.

Proof. In Section 4, we prove $R_J = R_{\mathrm{GGS}}$. On the other hand, it is known that R_{GGS} satisfies the QYBE in this case [8]. □

Remark 1.5 In fact, one may check $R_J = R_{\mathrm{GGS}}$ when τ is replaced by the map τ^k for $k \in \mathbb{Z}\backslash\{0\}$. Combining this with the result on generalized disjoint triples and a generalization of the union arguments in the following section, one may conclude that $R_J = R_{\mathrm{GGS}}$ whenever $\Gamma_1 = \bigcup \Gamma_1^{(i)}$ where $\Gamma_1^{(i)} \perp \Gamma_1^{(j)}, i \neq j$, and $\tau\Gamma_1^{(i)} \cap \Gamma_1^{(j)} = \emptyset$ whenever $i > j$. In particular, this includes the case when τ sends everything in the same direction–i.e., $\tau(\alpha_i) = \alpha_j$ implies $j > i$ for all i (or $i < j$ for all i). (The proof is omitted).

1.5 Maximal triples and unions

In this subsection, we summarize reductions of the twist and GGS conjectures which are proved in the following two subsections.

Definition 1.4 *[5] We say that* $(\Gamma_1', \Gamma_2', \tau') < (\Gamma_1, \Gamma_2, \tau)$ *if* $\Gamma_1' \subset \Gamma_1$ *and* $\tau' = \tau\big|_{\Gamma_1'}$.

The following theorem reduces the twist and GGS conjectures to the case of maximal triples.

Theorem 1.6 *Suppose* $(\Gamma_1', \Gamma_2', \tau') < (\Gamma_1, \Gamma_2, \tau)$ *are Belavin-Drinfeld triples. Then if the twist or GGS conjecture holds for the larger triple, it also holds for the smaller one.*

Proof. See Section 1.6. □

Definition 1.5 *Define* $(\Gamma_1, \Gamma_2, \tau) = \bigcup(\Gamma_1^{(i)}, \Gamma_2^{(i)}, \tau^{(i)})$ *by* $\Gamma_1 = \bigcup \Gamma_1^{(i)}, \Gamma_2 = \bigcup \Gamma_2^{(i)}$, *with* $\tau : \Gamma_1 \to \Gamma_2$ *given by* $\tau\big|_{\Gamma_1^{(i)}} = \tau^{(i)}$. *Call a union orthogonal if* $\Gamma_1^{(i)} \perp \Gamma_1^{(j)}$ *and* $\Gamma_2^{(i)} \perp \Gamma_2^{(j)}$. *Furthermore, an orthogonal union is termed* τ*-orthogonal if, in addition,* $\Gamma_2^{(i)} \cap \Gamma_1 \subset \Gamma_1^{(i)}, \forall i$.

It is easy to check that an orthogonal union of Belavin-Drinfeld triples always defines a Belavin-Drinfeld triple (one must check that τ is nilpotent and a graph isomorphism for the union). A triple that is a τ-orthogonal union of two nonempty triples is called *decomposable*; otherwise it is *indecomposable*. The following theorem reduces the twist and GGS conjectures to the case of indecomposable triples.

Theorem 1.7 *If* $(\Gamma_1, \Gamma_2, \tau)$ *is a* τ*-orthogonal union of* $(\Gamma_1^{(i)}, \Gamma_2^{(i)}, \tau^{(i)})$, *then the twist or GGS conjecture holds for the union iff it holds for each triple* $(\Gamma_1^{(i)}, \Gamma_2^{(i)}, \tau^{(i)})$.

Proof. See Section 1.7. □

1.6 Maximal triples

In this section, by Giaquinto's suggestion, we investigate the notion of maximal triples as defined in Definition 1.4 with the goal of proving Theorem 1.6. We will assume throughout this section that $(\Gamma_1', \Gamma_2', \tau') < (\Gamma_1, \Gamma_2, \tau)$. Define G, H, X for $(\Gamma_1, \Gamma_2, \tau)$ as in Section 2.1, 2.2 and similarly G', H', X' for $(\Gamma_1', \Gamma_2', \tau')$. We begin with an important result.

Proposition 1.6 *If $\sum_{k=1}^{m} (\tau\alpha_{i_k} - \alpha_{i_k}) = \sum_{k=1}^{m} (\tau\alpha_{j_k} - \alpha_{j_k})$ then $\exists \rho \in S_m$, a permutation, such that $i_k = j_{\rho(k)}, \forall k$.*

Proof. Suppose that $\nexists \rho \in S_m$ such that $i_k = j_{\rho(k)}, \forall k$. Then, let $\alpha_l \in \Gamma_1$ be a maximal simple root under the ordering $\prec$ so that $\tau\alpha_l - \alpha_l$ does not appear the same number of times in the sequences $(i_k), (j_k)$. Then, $0 = \sum_{k=1}^{m} (\tau\alpha_{j_k} - \tau\alpha_{i_k} - \alpha_{j_k} + \alpha_{i_k}) = p\tau\alpha_l + \sum_k \alpha_{o_k}$ where $p \neq 0$ and $\tau\alpha_l \neq \alpha_{o_k}$ for any k. This is a contradiction. $\square$

Define $H = \{\sum_{i=1}^{m} \tau\alpha_{k_i} - \alpha_{k_i} \mid \alpha_{k_i} \in \Gamma_1, m \geq 2\}$, and $G = \{e_\alpha \otimes e_\beta \mid \alpha + \beta \in H$. Clearly $(R_J)_+, (R_J)_- \in \mathrm{Span}_{\mathbb{C}[q,q^{-1}]}(G)$, and moreover, $G \subset \tilde{\Gamma}_1 \times (\tilde{\Gamma}_2)^T \cup (\tilde{\Gamma}_2)^T \times \tilde{\Gamma}_1$ where T takes the transpose of any element. Furthermore, let $V_0 \subset Mat_n(\mathbb{C}) \otimes Mat_n(\mathbb{C})$ be the space of zero weight in the representation $\mathfrak{g} \otimes \mathfrak{g}$ of $\mathfrak{g}$. Elements will be said to have zero weight. Then, we may define the following:

Definition 1.6 *For any element $x \in H$, define $i_\tau(x) = m$ if $x = \sum_{k=1}^{m} (\tau\alpha_{i_k} - \alpha_{i_k})$. m is called the τ-index of x. Similarly define $i_\tau(e_\alpha \otimes e_\beta) \equiv i_\tau(\alpha + \beta)$ when $e_\alpha \otimes e_\beta \in G$.*

Clearly $i_\tau(x + y) = i_\tau(x) + i_\tau(y)$ for $x, y \in H$ and $i_\tau(xy) = i_\tau(x) + i_\tau(y)$ for $x, y \in G$ and $xy \neq 0$. Note that if $|\tau^k\alpha| = |\alpha|$, then $i_\tau(\tau^k\alpha - \alpha) = k|\alpha|$. This concept of index over H and G will come in handy in section 5.

Proposition 1.7 *Suppose $(\alpha, \beta) \in X \setminus X'$ and $y \in G$. Then $\{e_\beta \otimes e_{-\alpha}, e_{-\alpha} \otimes e_\beta, (e_\beta \otimes e_{-\alpha})y, (e_{-\alpha} \otimes e_\beta)y, y(e_\beta \otimes e_{-\alpha}), y(e_{-\alpha} \otimes e_\beta\} \cap G' = \emptyset$.*

Proof. Since $(\alpha, \beta) \notin X'$, it follows that $\beta - \alpha \notin H'$, and hence $\beta - \alpha + H \cap H' = \emptyset$. $\square$

Denote by $K_{\alpha,\beta}$ and $K'_{\alpha,\beta}$ the appropriate K-coefficients for the two triples, and by ϵ and ϵ' the appropriate ϵ-matrices.

Corollary 1.1 *(i) If $(\alpha, \beta) \in X'$, then $K_{\alpha,\beta} = K'_{\alpha,\beta}$.*
Now suppose $e_\alpha \otimes e_\beta \in G'$. Then (ii) $\epsilon_{\alpha,\beta} = \epsilon'_{\alpha,\beta}$, hence
(iii) $(R_J)_{\alpha,\beta} = (R'_J)_{\alpha,\beta}$, (iv) $(R_J)_{-\beta,-\alpha} = (R'_J)_{-\beta,-\alpha}$,
(v) $(R_{GGS})_{\alpha,\beta} = (R'_{GGS})_{\alpha,\beta}$, and (vi) $(R_{GGS})_{-\beta,-\alpha} = (R'_{GGS})_{-\beta,-\alpha}$.

Proof. The proposition shows that terms from $G \setminus G'$ do not affect terms in G' when expanding (2.1) and the matrices $\epsilon, R_J, R_{\mathrm{GGS}}$. Note that this result also follows from the combinatorial formulas (1.5), (1.11) (which are derived independently). $\square$

Proof of Theorem 1.6. (i) Choose r^0 to satisfy equation (1.1) for the triple $(\Gamma_1, \Gamma_2, \tau)$. Clearly the equations for $(\Gamma_1', \Gamma_2', \tau)$ are a subset of these, and by Remark 1.2, it is sufficient to consider only this r^0. With this r^0, define $\bar{R}_J, \bar{R}_{\mathrm{GGS}}$ corresponding to $(\Gamma_1, \Gamma_2, \tau)$ and $\bar{R}'_J, \bar{R}'_{\mathrm{GGS}}$ corresponding to $(\Gamma_1', \Gamma_2', \tau)$.

Construct $f \in \mathfrak{h}$ as follows:

$$\tau\alpha_k f = \alpha_k f, \quad \alpha_k \in \Gamma_1 \setminus \Gamma_1', \tag{1.12}$$
$$\tau\alpha_k f = 1 + \alpha_k f, \alpha_k \in \Gamma_1'. \tag{1.13}$$

Then, we see that $e^{ft} \otimes e^{ft} e_{\tau\alpha} \otimes e_{-\alpha} e^{-ft} \otimes e^{-ft} = e^t e_{\tau\alpha} \otimes e_{-\alpha}$ whenever $\alpha \in \Gamma_1 \setminus \Gamma_1'$, and $e_{\tau\alpha} \otimes e_{-\alpha}$ otherwise (that is, when $\alpha \in \tilde{\Gamma}_1'$). This obviously holds as well for $e_{-\alpha} \otimes e_{\tau\alpha}$. Clearly, if one takes any term x of zero weight, congugation by $e^{ft} \otimes e^{ft}$ leaves the term unaltered. This implies that $(e^{ft} \otimes e^{ft}) x (e^{-ft} \otimes e^{-ft}) = e^{kt} x$ for $k \in Z^+$ whenever $x \in G \setminus G'$, but $k = 0$ when $x \in G' \cup V_0$. It follows from Proposition 1.7 that $\lim_{t \to -\infty} e^{ft} \otimes e^{ft} \bar{R}_J e^{-ft} \otimes e^{-ft} = \bar{R}_J'$ and $\lim_{t \to -\infty} e^{ft} \otimes e^{ft} \bar{R}_{\mathrm{GGS}} e^{-ft} \otimes e^{-ft} = \bar{R}_{\mathrm{GGS}}'$. This clearly implies the theorem. □

1.7 Unions of triples

In this section, we investigate unions as defined in Definition 1.5 with the goal of proving Theorem 1.7. We will see in Lemma 1.2, in fact, that the matrices R_{GGS} and R_J for the τ-orthogonal union follows directly from those for each triple.

Since it is clear that a union of triples is larger than each piece under the ordering of the previous sections, we may pick r^0 to satisfy (1.1) for the union which includes all equations for each smaller triple. Fix some such r^0, which will be sufficient by Remark 1.2 to make statements about the conjectures. We use the notation $R_{\mathrm{GGS}}^{(i)}$ and $R_J^{(i)}$ for the respective R matrices. Set $R_s' \equiv q^{\tilde{r}^0} R_s q^{\tilde{r}^0}$. In addition, set $S_b^a \equiv R_b^a - R_s'$ for any such subscripts b and superscripts a, and similarly $\bar{S}_b^a \equiv \bar{R}_b^a - R_s$. We will use H, G as defined in Section 2.2, and define $H^{(i)}, G^{(i)}$ for the respective subtriples. Finally, define $V_k^{(i)} = \{e_\alpha \mid \alpha \in \tilde{\Gamma}_k^{(i)} \cup \tilde{\Gamma}_2^{(i)}\}$ for $k \in \{1, 2\}$, and let $V^{(i)} = \{e_\alpha \mid \alpha \in (\tilde{\Gamma}_1 + \tilde{\Gamma}_2) \cap \Gamma\}$.

Lemma 1.1 *Suppose that $(\Gamma_1, \Gamma_2, \tau)$ is a τ-orthogonal union of $(\Gamma_1^{(i)}, \Gamma_2^{(i)})$. Then $V_k^{(i)} V_k^{(j)} = \emptyset$ for $i \neq j, k \in \{1, 2\}$, and $V_1^{(i)} (V_2^{(j)})^T = (V_2^{(j)})^T V_1^{(i)} = \emptyset$ for $i \neq j$, where T takes the transpose of each element. Hence one has $(G^{(i)})_{ab} X (G^{(j)})_{cd} = \emptyset$ for $a, b, c, d \in \{1, 2, 3\}, a \neq b, c \neq d$ and any $X \in V_0 \otimes V_0 \otimes V_0$.*

Proof. Indeed, the first two assertions follow from the facts $\Gamma_1^{(i)} \perp \Gamma_1^{(j)}, i \neq j$, and $\Gamma_1^{(i)} \cap \Gamma_2^{(j)} = \emptyset, i \neq j$. The remainder follows since $G_b^a V_0, V_0 G_b^a \subset \mathrm{Span}(G_b^a)$. □

Lemma 1.2 *Suppose that $(\Gamma_1, \Gamma_2, \tau)$ is a τ-orthogonal union of $(\Gamma_1^{(i)}, \Gamma_2^{(i)})$. Then $S_t = \sum_i S_t^{(i)}$ for $t \in \{GGS, J\}$.*

Proof. Clearly any element of G is of the form $(e_\beta \otimes e_{-\alpha})_{ab}$ for $\{a, b\} = \{1, 2\}$ and $\alpha \in \tilde{\Gamma}_1, \beta \in \tilde{\Gamma}_2$. It is clear that $\tilde{\Gamma}_1 = \sqcup_i \tilde{\Gamma}_1^{(i)}$ where $\sqcup$ denotes a disjoint union. Since we also have $\tilde{\Gamma}_2^{(i)} \cap \tilde{\Gamma}_1^{(j)} = 0$ for $i \neq j$, it follows that $G = \sqcup_i G^{(i)}$. Hence, $S_t = \sum_i S_t^{(i)}$ for $t \in \{\mathrm{GGS}, J\}$. □

Theorem 1.8 *If $(\Gamma_1, \Gamma_2, \tau)$ is a τ-orthogonal union of $(\Gamma_1^{(i)}, \Gamma_2^{(i)})$, then any part of the twist or GGS conjecture holds for each triple $(\Gamma_1^{(i)}, \Gamma_2^{(i)}, \tau^{(i)})$ iff the same part of the twist or GGS conjecture holds for the union.*

Proof. By Lemma 1.1, we have $(S_t^{(i)})_{ab}X(S_t^{(j)})_{cd} = 0$ for $a,b,c,d \in \{1,2,3\}, a \neq b, c \neq d$, and any $X \in \mathrm{End}(Mat_n(\mathbb{C}) \otimes Mat_n(\mathbb{C}) \otimes Mat_n(\mathbb{C}))$, one may consider the QYBE separately in $V_0 \otimes V_0 \otimes V_0$ and in $V^{(i)} \otimes V^{(i)} \otimes V^{(i)}$ for each i, and one may consider the Hecke relation and $R_J = R_{\mathrm{GGS}}$ separately in $V_0 \otimes V_0, V^{(i)} \otimes V^{(i)}$. The V_0 components clearly only involve R_s so are satisfied, while each $V^{(i)}$ component holds iff the respective equation holds for the i-th subtriple. Finally, the respective equation holds for the union iff it holds in each component, and each component yields the same equation considered in the union and in the appropriate subtriple by Lemma 1.2. The theorem follows from these observations. ☐

2 Complete description of $K_{\alpha,\beta}$

In this section we prove Proposition 1.3 and Theorem 1.2 by first deducing (1.11) from $\frac{d^2}{d\hbar^2}\big|_{\hbar=0}\bar{R}_{\mathrm{GGS}}$ and then applying the development to prove the equivalence of (1.3) and (1.5) for ϵ. To do this, we will rely on important bijections between different ways the product of two terms in aP, Pa, a^2 can arise, where corresponding pairs cancel in the expansion.

Lemma 2.1 *There exist unique $K_{\alpha,\beta}$ such that $\frac{d^2}{d\hbar^2}(R_J - (R_J)_{21})_{\alpha,\beta} = 0, \alpha \prec \beta$. These are given by the formula*

$$K_{\alpha,\beta} = sign(\alpha,\beta)\Big[\sum_{i \geq j} a_+^i a_+^j - \sum_{i<j} a_+^i a_+^j + a_+P_- + a_-P_+ + P_+a_+ + a_+a_- - a_-a_+\Big]_{\alpha,\beta}. \tag{2.1}$$

With these $K_{\alpha,\beta}$, the condition that $R_J \equiv R_{GGS} \pmod{\hbar^3}$ reduces to showing the equivalence of (2.1) and (1.11), and that $K_{\alpha,\beta} = \epsilon_{\alpha,\beta} = 0$ when $\alpha \not\prec \beta$, in this case defining $K_{\alpha,\beta}$ by (2.1) for all $\alpha,\beta \in \tilde{\Gamma}$.

Proof. We expand $\bar{R}_J$ modulo $\hbar^3$ as follows:

$$\begin{aligned}\bar{R}_J \equiv \Big[\prod_{(\alpha,\beta)\in X}^{>}\big(1 - 2\,\mathrm{sign}(\alpha,\beta)\hbar(1+K_{\alpha,\beta}\hbar)\, e_\beta \otimes e_{-\alpha}\big)\Big]\Big[1 + \frac{\hbar^2}{2}\sum_i e_{ii}\otimes e_{ii} + 2\hbar P_-\Big] \\ \Big[\prod_{(\alpha,\beta)\in X}^{<}\big(1 + 2\,\mathrm{sign}(\alpha,\beta)\hbar(1+K_{\alpha,\beta}\hbar)\, e_{-\alpha}\otimes e_\beta\big)\Big] \equiv 1 + 2\hbar\Big[P_- + \sum_{\alpha\prec\beta}\mathrm{sign}(\alpha,\beta)\, e_{-\alpha}\wedge e_\beta\Big] \\ + \hbar^2\Big[\frac{1}{2}\sum_i e_{ii}\otimes e_{ii} + 2\sum_{\alpha\prec\beta}K_{\alpha,\beta}\,\mathrm{sign}(\alpha,\beta)\, e_{-\alpha}\wedge e_\beta + 4a_+P_- + 4P_-a_- + 4a_+a_- \\ + 4\sum_{i\geq j}a_+^i a_+^j + 4\sum_{i<j}a_-^i a_-^j\Big] \pmod{\hbar^3}. \end{aligned}\tag{2.2}$$

If we skew-symmetrize the second order terms in (2.2) by the substitution $x \mapsto x - x_{21}$, (2.1) follows as a necessary and sufficient condition for $\frac{d^2}{\hbar^2}\big|_{\hbar=0}\big[\bar{R}_J - (\bar{R}_J)_{21}\big]_{\alpha,\beta} = 0$.

It is clear that $R_J \equiv R_{\mathrm{GGS}} \equiv 1 + 2\hbar r \pmod{\hbar^3}$, so in particular by the comments in Section 1.2, this implies $\frac{1}{2}(R_J + (R_J)_{21}) \equiv R_{\mathrm{GGS}} \equiv 1 + \hbar P + 2\hbar^2 s \pmod{\hbar^3}$. All that remains,

then, is to show that (1.11) and (2.1) are equivalent, and that $\epsilon_{\alpha,\beta} = \epsilon_{-\beta,-\alpha} = 0$ when $\alpha \not\prec \beta$ and the same for $K_{\alpha,\beta}$ in (2.1). $\square$

Take positive roots α, β, $|\alpha| = |\beta|, \alpha \neq \beta$. Set $\alpha = e_i - e_j$, $\beta = e_k - e_l$. Then, say that $\alpha < \beta$ if $i < k$, and in this case, $\alpha \lessdot \beta$ if $j = k$, $\alpha \overline{<} \beta$ if $j > k$, and $\alpha \ll \beta$ if $j < k$ (we repeat the definition of $\lessdot$ given in Section 1 for completeness.) We will use $\alpha > \beta$ if $\beta < \alpha$, and similarly, $\gtrdot, \overline{>}$, and $\gg$ are the reverse directions of $\lessdot, \overline{<}$, and $\ll$, respectively. With these definitions, we will take $x^{<} = \sum_{\alpha<\beta}\big(x_{\alpha,\beta}\, e_\beta \otimes e_{-\alpha} + x_{-\beta,-\alpha}\, e_{-\alpha} \otimes e_\beta\big)$ and similarly for the other defined relations.

Lemma 2.2 *We may rewrite* (2.1) *as follows:*

$$K_{\alpha,\beta} = sign(\alpha,\beta)\Bigg[\frac{1}{2}\big(a_+^{\gtrdot} - a_+^{\lessdot}\big) + a_+^{\overline{>}}P - Pa_+^{\overline{<}} + \sum_{\alpha' \prec^{\leftarrow} \beta'} sign(\alpha',\beta')(1-|\alpha'|)\, e_{\beta'} \otimes e_{-\alpha'} + \sum_{i<j}\big(a_+^j a_+^i - a_+^i a_+^j\big) + a_+a_- - a_-a_+\Bigg]_{\alpha,\beta}. \quad (2.3)$$

Proof. First, we expand $a_+P_- + a_-P_+ + P_+a_+$:

$$\big(a_+P_- + a_-P_+ + P_+a_+\big)_+ = \Big[a_+^{\overline{>}}P + \frac{1}{2}a_+^{\gtrdot} + a_-^{\leq}P + Pa_+^{\ll} + \frac{1}{2}a_+^{\lessdot}\Big]_+ = \Big[\frac{1}{2}\big(a_+^{\lessdot} + a_+^{\gtrdot}\big) + a_+^{\overline{>}}P - Pa_+^{<} + Pa_+^{\ll}\Big]_+ = \frac{1}{2}\big(a_+^{\gtrdot} - a_+^{\lessdot}\big) + a_+^{\overline{>}}P - Pa_+^{\overline{<}}. \quad (2.4)$$

Now we simplify $\sum_i (a_+^i)^2$. $\sum_i (a_+^i)^2 = \sum_{\alpha\prec\beta,\alpha'\prec\beta'} \mathrm{sign}(\alpha,\beta)\mathrm{sign}(\alpha',\beta')\, e_\beta e_{\beta'} \otimes e_{-\alpha}e_{-\alpha'}$. For $\alpha, \alpha' \in \tilde{\Gamma}_1$, $\tau^i(\alpha) = \beta, \tau^i(\alpha') = \beta'$, one sees that $e_\beta e_{\beta'} = e_{\beta+\beta'}$ and $e_{-\alpha}e_{-\alpha'} = e_{-\alpha-\alpha'}$ iff $\alpha + \alpha' \in \tilde{\Gamma}_1$ and $\tau^i(\alpha+\alpha') = \beta + \beta'$, reversing order. Thus, since in this case $\mathrm{sign}(\alpha,\beta)\mathrm{sign}(\alpha',\beta') = -\mathrm{sign}(\alpha+\alpha',\beta+\beta')$,

$$\sum_i (a_+^i)^2 = -\sum_{\alpha\prec^{\leftarrow}\beta} \mathrm{sign}(\alpha,\beta)(|\alpha|-1)\, e_\beta \otimes e_{-\alpha}. \quad (2.5)$$

It is clear that (2.4) and (2.5) imply the proposition. $\square$

Now, we proceed to show the equivalence of (2.3) and (1.11) by canceling most terms in the expansion of (2.1) pairwise. Define the following sets:

$$M_1 = \{((\alpha,\tau^x\alpha),(\beta,\tau^y\beta)) \in X \times X \mid \alpha \gtrdot \beta, \tau^x\alpha \lessdot \tau^y\beta, x > y\}, \quad (2.6)$$
$$M_2 = \{((\alpha,\tau^x\alpha),(\beta,\tau^y\beta)) \in X \times X \mid \alpha \gtrdot \beta, \tau^x\alpha \lessdot \tau^y\beta, x < y\}, \quad (2.7)$$
$$M_3 = \{((e_x - e_y, e_u - e_v),(e_{v'} - e_v, e_x - e_{x'})) \in X \times X \mid x' < y, u < v'\}, \quad (2.8)$$
$$M_4 = \{((e_x - e_y, e_u - e_v),(e_u - e_{u'}, e_{y'} - e_y)) \in X \times X \mid x < y', u' < v\}, \quad (2.9)$$
$$M_5 = \{(\alpha,\beta) \in X \mid \alpha \overline{>} \beta\}, \quad (2.10)$$
$$M_6 = \{(\alpha,\beta) \in X \mid \alpha \overline{<} \beta\}. \quad (2.11)$$

Clearly these are defined so that the following hold:

$$\sum_{i<j} a_+^j a_+^i = \sum_{((\alpha,\beta),(\gamma,\delta))\in M_1} a_{\alpha,\beta} a_{\gamma,\delta} e_{\beta+\delta} \otimes e_{-\alpha-\delta}, \tag{2.12}$$

$$\sum_{i<j} a_+^i a_+^j = \sum_{((\alpha,\beta),(\gamma,\delta))\in M_2} a_{\alpha,\beta} a_{\gamma,\delta} e_{\beta+\delta} \otimes e_{-\alpha-\delta}, \tag{2.13}$$

$$(a_+ a_-)_+ = \sum_{((\alpha,\beta),(\gamma,\delta))\in M_3} a_{\alpha,\beta} a_{-\delta,-\gamma} e_{\alpha-\delta} \otimes e_{\beta-\gamma}, \tag{2.14}$$

$$(a_- a_+)_+ = \sum_{((\alpha,\beta),(\gamma,\delta))\in M_4} a_{-\delta,-\gamma} a_{\alpha,\beta} e_{\alpha-\delta} \otimes e_{\beta-\gamma}, \tag{2.15}$$

$$a_+^{\gtrdot} P = \sum_{(\alpha,\beta)\in M_5} a_{\alpha,\beta} e_\beta \otimes e_{-\alpha} P, \quad P a_+^{\lessdot} = \sum_{\alpha,\beta\in M_6} a_{\alpha,\beta} P e_\beta \otimes e_{-\alpha}. \tag{2.16}$$

Now we define subsets $M_i' \subset M_i$, set $M_i'' = M_i \setminus M_i'$, and bijections $f : M_1' \to M_3', g : M_2' \to M_4', f' : M_1'' \to M_5', g' : M_2'' \to M_6'$ which allow pairwise cancellation, leaving us to expand (2.3) by only those terms in $M_3'', M_4'', M_5'', M_6''$, which will lead directly to (1.11). Define M_i' as follows:

$$M_1' = \{((\alpha,\tau^x\alpha),(\beta,\tau^y\beta)) \in M_1 \mid x-y \nmid y\}, \tag{2.17}$$

$$M_2' = \{((\alpha,\tau^x\alpha),(\beta,\tau^y\beta)) \in M_2 \mid y-x \nmid x\}, \tag{2.18}$$

$$M_3' = \{((e_x - e_y, e_u - e_v),(e_{v'} - e_v, e_x - e_{x'})) \in M_3 \mid (y-x') \nmid (y-x)\}, \tag{2.19}$$

$$M_4' = \{((e_x - e_y, e_u - e_v),(e_u - e_{u'}, e_{y'} - e_y)) \in M_4 \mid (y'-x) \nmid (y-x)\}, \tag{2.20}$$

$$M_5' = \{(e_x - e_y, e_u - e_v) \in M_5 \mid x-u \nmid y-x\}, \tag{2.21}$$

$$M_6' = \{(e_x - e_y, e_u - e_v) \in M_6 \mid u-x \nmid y-x\}. \tag{2.22}$$

Now, we construct bijections f, g, f', g'. We begin with f. Take $((\alpha,\tau^x\alpha),(\beta,\tau^y\beta)) \in M_1'$. Suppose $y = p(x-y)+q$ where $p,q \in \mathbb{N}$ and $0 < q < x-y$. Then $\alpha \gtrdot \beta \gtrdot \tau^{x-y}(\alpha+\beta) \gtrdot \cdots \gtrdot \tau^{p(x-y)}(\alpha+\beta) \gtrdot \tau^{(p+1)(x-y)}\alpha$. Then,

$$\begin{aligned} f((\alpha,\tau^x\alpha),(\beta,\tau^y\beta)) = \Big[&\big((1+\tau^{x-y}+\ldots+\tau^{p(x-y)})(\alpha+\beta)+\tau^{(p+1)(x-y)}\alpha, \\ &(\tau^q+\tau^{q+(x-y)}+\ldots+\tau^y)(\alpha+\beta)+\tau^x\alpha\big), \big((\tau^q+\ldots+\tau^{q+(p-1)(x-y)})(\alpha+\beta)+\tau^y\alpha, \\ &(\tau^{x-y}+\ldots+\tau^{p(x-y)})(\alpha+\beta)+\tau^{(p+1)(x-y)}\alpha\big)\Big] \in M_3'. \end{aligned} \tag{2.23}$$

Similarly, if $((\alpha,\tau^x\alpha),(\beta,\tau^y\beta)) \in M_2'$, one sets $x = p(y-x)+q$, $0 < q < x-y$, notices $\beta \lessdot \alpha \lessdot \tau^{y-x}(\alpha+\beta) \lessdot \cdots \lessdot \tau^{p(y-x)}(\alpha+\beta) \lessdot \tau^{(p+1)(x-y)}\beta$, and is able to define

$$\begin{aligned} g((\alpha,\tau^x\alpha),(\beta,\tau^y\beta)) = \Big[&\big((1+\tau^{y-x}+\ldots+\tau^{p(y-x)})(\alpha+\beta)+\tau^{(p+1)(y-x)}\beta, \\ &(\tau^q+\tau^{q+(y-x)}+\ldots+\tau^x)(\alpha+\beta)+\tau^y\beta\big), \big((\tau^q+\ldots+\tau^{q+(p-1)(y-x)})(\alpha+\beta)+\tau^x\beta, \\ &(\tau^{y-x}+\ldots+\tau^{p(y-x)})(\alpha+\beta)+\tau^{(p+1)(y-x)}\beta\big)\Big] \in M_4'. \end{aligned} \tag{2.24}$$

Next, we define f' and g':

$$f'((e_j - e_i, e_a - e_{a+i-j}), (e_k - e_j, e_{a+i-j} - e_{a+i-k})) = (e_{a+i-k} - e_i, e_a - e_k) \in M_5'', \quad (2.25)$$
$$g'((e_j - e_i, e_a - e_{a+i-j}), (e_k - e_j, e_{a+i-j} - e_{a+i-k})) = (e_k - e_a, e_i - e_{a+i-k}) \in M_6''. \quad (2.26)$$

Lemma 2.3 *(i) $f : M_1' \to M_3'$ is bijective. Given any $((\alpha,\beta),(\gamma,\delta)) \in M_1'$, and $f((\alpha,\beta),(\gamma,\delta)) = ((\alpha',\beta'),(\gamma',\delta')) \in M_3'$, one has $a_{\alpha,\beta}a_{\gamma,\delta} + a_{\alpha',\beta'}a_{-\delta',-\gamma'} = 0$.*

(ii) $g : M_2' \to M_4'$ is bijective. Given any $((\alpha,\beta),(\gamma,\delta)) \in M_2'$, and $g((\alpha,\beta),(\gamma,\delta)) = ((\alpha',\beta'),(\gamma',\delta')) \in M_4'$, one has $a_{\alpha,\beta}a_{\gamma,\delta} + a_{\alpha',\beta'}a_{-\delta',-\gamma'} = 0$.

(iii) $f' : M_1'' \to M_5'$ is bijective. Given any $((\alpha,\beta),(\gamma,\delta)) \in M_1''$, and $f'((\alpha,\beta),(\gamma,\delta)) = (\alpha',\beta') \in M_5'$, one has $a_{\alpha,\beta}a_{\gamma,\delta} + a_{\alpha',\beta'} = 0$.

(iv) $g' : M_2'' \to M_6'$ is bijective. Given any $((\alpha,\beta),(\gamma,\delta)) \in M_2''$, and $g'((\alpha,\beta),(\gamma,\delta)) = (\alpha',\beta') \in M_5'$, one has $a_{\alpha,\beta}a_{\gamma,\delta} + a_{\alpha',\beta'} = 0$.

Proof. (i) Take any $((e_x - e_y, e_u - e_v),(e_{v'} - e_v, e_x - e_{x'})) \in M_3'$. We find its inverse under f and verify the identity. Suppose $\tau^r(e_x - e_y) = e_u - e_v, \tau^s(e_{v'} - e_v) = e_x - e_{x'}$. Clearly τ^r preserves orientation on $e_x - e_y$.

Suppose τ^s reverses orientation on $e_{v'} - e_v$. In this case, nilpotency of τ shows that $x' - x \le y - x'$, so $y - x' \nmid x' - x$ implies $x' - x < y - x'$. Then, one sees that $\tau^{2r+s}(e_{y-(x'-x)} - e_y) = e_u - e_{u+(x'-x)}$, while $\tau^r(e_x' - e_{y-(x'-x)}) = e_{u+(x'-x)} - e_{u+(y-x')}$. It is easy to check that $f((e_{y-(x'-x)} - e_y, e_u - e_{u+(x'-x)}),(e_{x'-e_{y-(x'-x)}}, e_{u+(x'-x)} - e_{u+(y-x')})) = ((e_x - e_y, e_u - e_v),(e_{v'} - e_v, e_x - e_{x'}))$ as desired. Furthermore, we see that τ^{2r+s}, τ^s reverse orientation while τ^r preserves orientation, so the desired identity follows.

Now, suppose τ^s preserves orientation on $e_{v'} - e_v$. Then, $\tau^{r+s}(\alpha_i) = \alpha_{i-(y-x')}$ for $x + y - x' \le i \le y$. Then, suppose $x' - x = p(y - x') + q$, $0 < q < y - x'$. In this case, $\tau^{(p+1)(r+s)+r}(e_{y-q} - e_y) = (e_u - e_{u+q})$ and $\tau^{p(r+s)+r}(e_{x'} - e_{y-q}) = (e_{u+q} - e_{v'})$. One may check $f((e_{y-q} - e_y, e_u - e_{u+q}),(e_{x'} - e_{y-q}, e_{u+q} - e_{v'})) = ((e_x - e_y, e_u - e_v),(e_{v'} - e_v, e_x - e_{x'}))$, as desired. Since τ^r, τ^s both preserve orientation, the desired identity follows.

(ii) This follows exactly as in (i).

(iii) Take any $(e_x - e_y, e_u - e_v) \in M_5'$. We find its inverse under f' and verify the identity. Indeed, suppose $\tau^r(e_x - e_y) = e_u - e_v$ and $y - x = p(x - u) + q$ for $0 < q < x - u$. Then, $\tau^{(p+1)r}(e_{y-q} - e_y) = e_u - e_{u+q}$, and $\tau^{pr}(e_v - e_{y-q}) = e_{u+q} - e_x$, so that $f'((e_{y-q} - e_y),(e_u - e_{u+q})) = ((e_v - e_{y-q}),(e_{u+q} - e_x))$. Furthermore, τ^r preserves orientation on $e_x - e_y$ so the identity is verified.

(iv) This follows exactly as in (iii). $\square$

Proposition 2.1 *(i) Formula* (2.3) *is equivalent to* (1.11). *(ii) Formula* (1.3) *for ϵ is equivalent to* (1.5).

Proof. (i) Given any $\tau^z\alpha = \beta$, it is clear that $\exists((\gamma,\delta),(\gamma',\delta')) \in M_3''$ with $\gamma - \delta' = \alpha, \delta - \gamma' = \beta$ iff $\exists t, 0 < t < z, t \nmid z$ such that $\tau^t\alpha \lessdot \alpha$. In this case it is easy to see (along similar lines as (i) in the proof of Lemma 2.3) that $a_{\gamma,\delta}a_{-\delta',-\gamma'} = -\mathrm{sign}(\gamma',\delta') = -\mathrm{sign}(\alpha,\beta)$. Similarly, given any $\tau^z\alpha = \beta$, $\exists((\gamma,\delta),(\gamma',\delta')) \in M_4''$ such that $\gamma' - \delta = \alpha, \delta' - \gamma = \beta$ iff $\exists t, 0 < t < z, t \nmid z$ such that $\tau^t\alpha \gtrdot \alpha$. In this case, $a_{-\delta,-\gamma}a_{\gamma',\delta'} = -\mathrm{sign}(\gamma,\delta) = -\mathrm{sign}(\alpha,\beta)$. Next, we find that

$M_5'' = \{(\alpha, \tau^{tk}\alpha) \mid t,k \in \mathbb{Z}^+, \tau^t\alpha \lessdot \beta\}$ and $M_6'' = \{(\alpha, \tau^{tk}\alpha) \mid t,k \in \mathbb{Z}^+, \tau^t\alpha \gtrdot \beta\}$, and it is clear that all terms in $a_+^{\gtrless}, a_+^{\lessgtr}$ appear with coefficient -1. Hence, combining Lemma 2.3 with (2.3), we obtain precisely (1.11).

(ii) Indeed, using $r_s = P + P_- - P_+$ we find $\epsilon_+ = a_+a_+ + a_+a_- + a_-a_+ + Pa_+^{\lessgtr} + a_+^{\gtrless}P + \frac{1}{2}\big(a_+^{\lessdot} + a_+^{\gtrdot}\big)$. Then, we see as in Lemma 2.2 that $\sum_i (a_+^i)^2 = \sum_{\alpha \prec^{\leftarrow} \beta}(1-|\alpha|)\mathrm{sign}(\alpha,\beta)$, so $(a_+a_+)_{\alpha,\beta} = [\alpha \prec^{\leftarrow} \beta](-1)^{1-|\alpha|}(1-|\alpha|) + \sum_{i<j}(a_+^i a_+^j + a_+^j a_+^i)$. Hence, (1.5) follows from the observations in (i). $\square$

Proposition 1.3 and Theorem 1.2 are proved. $\square$

3 The disjoint case and its generalization

3.1 The disjoint case

This section is devoted to proving the following theorem:

Theorem 3.1 *If $\Gamma_1 \cap \Gamma_2 = \emptyset$, then $R_{GGS} = R_J$.*

Proof. We will assume $\Gamma_1 \cap \Gamma_2 = \emptyset$ throughout this section. The first observation to make in this case is that, since $\tau^2 = 0$, $J = J^1 = 1 + A^1 = 1 + (q - q^{-1})\sum_{\alpha \prec \beta} \mathrm{sign}(\alpha,\beta) q^{K_{\alpha,\beta}} e_\beta \otimes e_{-\alpha}$. Set $A = A^1$.

Let $\alpha \perp \beta$ denote either $\alpha \lll \beta$ or $\beta \lll \alpha$ (this is **not** the same as $(\alpha,\beta) = 0$). Using (1.11), the form of $K_{\alpha,\beta}$ in the disjoint case is summarized in the following table:

$K_{\alpha,\beta}$	$\alpha \prec^{\rightarrow} \beta$	$\alpha \prec^{\leftarrow} \beta$
$\alpha \perp \beta$	0	$1-\|\alpha\|$
$\alpha \lessdot \beta$	$\frac{1}{2}$	$\frac{3}{2}-\|\alpha\|$
$\alpha \gtrdot \beta$	$-\frac{1}{2}$	$\frac{1}{2}-\|\alpha\|$

Table 4.1: $K_{\alpha,\beta}$ in the disjoint case.

Also, ϵ is summarized as follows:

$\epsilon_{\alpha,\beta}$	$\alpha \prec^{\rightarrow} \beta$	$\alpha \prec^{\leftarrow} \beta$
$\alpha \perp \beta$	0	$(-1)^{1-\|\alpha\|}(1-\|\alpha\|)$
$\alpha \lessdot \beta$	$-\frac{1}{2}$	$(-1)^{1-\|\alpha\|}(\frac{1}{2}-\|\alpha\|)$
$\alpha \gtrdot \beta$	$-\frac{1}{2}$	$(-1)^{1-\|\alpha\|}(\frac{1}{2}-\|\alpha\|)$

Table 4.2: ϵ in the disjoint case.

Set $B = J^{-1} - 1$. Then, the following lemma describes $\bar{R}_J$:

Lemma 3.1 *(i) B is given by a sum $\sum_{\alpha \in \tilde{\Gamma}_1} B_{\alpha,\tau\alpha}\, e_{\tau\alpha} \otimes e_{-\alpha}$.*

(ii) $\bar{R}_J$ is given by the following equation:

$$\bar{R}_J = R_s + B + A_{21} + (q-1)B^{\gtrdot} + (q^{-1}-1)A_{21}^{\lessdot} \tag{3.1}$$

Proof. (i) Note that $e_{-\alpha}e_{-\beta} = e_{-\alpha-\beta}$, $e_{\tau\alpha}e_{\tau\beta} = e_{\tau\alpha+\tau\beta} = e_{\tau(\alpha+\beta)}$, for $\alpha, \beta \in \tilde{\Gamma}_1$. Thus, when B is expanded, all terms will remain of this type.

(ii) It is clear that $\bar{R}_J = (1+B)R_s(1+A_{21}) = R_s + BR_s + R_sA_{21} + BR_sA_{21}$. Since $e_\beta e_{-\alpha} = e_{-\alpha}e_\beta = 0$ for all $\beta \in \tilde{\Gamma}_2, \alpha \in \tilde{\Gamma}_1$, we see that $BR_sA_{21} = (q-q^{-1})B\big(\sum_{\alpha>0} e_{-\alpha} \otimes e_\alpha\big)A_{21}$. Also, $B\big(\sum_{\alpha>0} e_{-\alpha} \otimes e_\alpha\big) = PB_{21}^{<}$ and $\big(\sum_{\alpha>0} e_{-\alpha} \otimes e_\alpha\big)A_{21} = P(A_{21}^{<} - A_{21}^{\lessdot})$. So,

$$\bar{R}_J = R_s + BR_s + R_sA_{21} + (q-q^{-1})PB_{21}^{<}A_{21}$$
$$= R_s + (q-q^{-1})P\left(B_{21}^{<} + A_{21}^{<} - A_{21}^{\lessdot} + B_{21}^{<}A_{21}^{<}\right) + B + A_{21} + (q-1)(B^{\gtrdot} + A_{21}^{\lessdot}).$$

Since $B_{21}^{<} + A_{21}^{<} + B_{21}^{<}A_{21}^{<} = (1 + A_{21}^{<})(1 + B_{21}^{<}) - 1 = 0$, we find:

$$\bar{R}_J = R_s - (q-q^{-1})PA_{21}^{\lessdot} + B + A_{21} + (q-1)(B^{\gtrdot} + A_{21}^{\lessdot})$$
$$= R_s + B + A_{21} + (q-1)B^{\gtrdot} + (q^{-1}-1)A_{21}^{\lessdot}.$$

The lemma is proved. $\square$

Now, we compute B using (1.10), in which $(\alpha, \beta) > (\alpha', \beta')$ whenever $(e_\beta \otimes e_{-\alpha})(e_{\beta'} \otimes e_{-\alpha'}) \neq 0$. Define $L_{\alpha,\beta}$ as follows:

$$L_{\alpha,\beta} = \begin{cases} 0 & \text{if } \alpha \perp \beta, \\ \frac{1}{2} & \text{if } \alpha \lessdot \beta, \\ -\frac{1}{2} & \text{if } \alpha \gtrdot \beta. \end{cases} \tag{3.2}$$

Lemma 3.2 *(i) If $\alpha \prec^{\rightarrow} \beta$, then $B_{\alpha,\beta} = -A_{\alpha,\beta} = -(q-q^{-1})q^{K_{\alpha,\beta}}$.*

(ii) If $\alpha \prec^{\leftarrow} \beta$, then $B_{\alpha,\beta} = -sign(\alpha,\beta)(q-q^{-1})q^{|\alpha|-1+L_{\alpha,\beta}}$.

Proof. (i) Clearly, if $\alpha \prec^{\rightarrow} \beta$, then $B_{\alpha,\beta} = -(q-q^{-1})q^{K_{\alpha,\beta}}$ since $(e_{\beta'} \otimes e_{-\alpha'})(e_{\beta''} \otimes e_{-\alpha''}) \neq 0$ only if $\alpha' \prec^{\leftarrow} \beta'$, $\alpha'' \prec^{\leftarrow} \beta''$, and in this case $\alpha' + \alpha'' \prec^{\leftarrow} \beta' + \beta''$.

(ii) We prove the lemma inductively. If $|\alpha| = 1$, (ii) is clear. Otherwise, assume (ii) holds for $|\alpha| \leq p$. We will prove the result for $|\alpha| = p+1$.

Suppose $\alpha = e_i - e_{i+p+1}$, $\beta = e_j - e_{j+p+1}$, and $\tau(\alpha_{i+k}) = \alpha_{j+p-k}$, $0 \leq k \leq p$. Then, by (1.10), we may write

$$B_{\alpha,\beta} = -\text{sign}(\alpha,\beta)(q-q^{-1})q^{K_{\alpha,\beta}} - \sum_{l=1}^{p} A_{e_i - e_{i+l}, e_{j+p+1-l} - e_{j+p+1}} B_{e_{i+l} - e_{i+p+1}, e_j - e_{j+p+1-l}}$$
$$= -\text{sign}(\alpha,\beta)(q-q^{-1})q^{K_{\alpha,\beta}} - \sum_{l=1}^{p} \text{sign}(\alpha,\beta)(q-q^{-1})^2 q^{p+1-2l+L_{\alpha,\beta}}$$
$$= -\text{sign}(\alpha,\beta)(q-q^{-1})q^{L_{\alpha,\beta}}\left[q^{-p} + \sum_{l=1}^{p}(q-q^{-1})q^{p-2l+1}\right] = -\text{sign}(\alpha,\beta)(q-q^{-1})q^{p+L_{\alpha,\beta}}. \quad \square$$

Using $e_{-\alpha} \wedge_c e_\beta$ as in Example 1.1, (3.1) becomes

$$\bar{R}_J = R_s + (q - q^{-1})\Big[\sum_{\alpha \prec^{\rightarrow} \beta} e_{-\alpha} \wedge_{-\frac{1}{2}(\alpha,\beta)} e_\beta + \sum_{\alpha \prec^{\leftarrow} \beta} (-1)^{|\alpha|-1} e_{-\alpha} \wedge_{-\frac{1}{2}(\alpha,\beta)+|\alpha|-1} e_\beta\Big]. \qquad (3.3)$$

All that remains is to show equivalence of (3.3) $\bar{R}_{\mathrm{GGS}}$. First we write $\bar{R}_{\mathrm{GGS}}$:

$$\bar{R}_{\mathrm{GGS}} = R_s + (q - q^{-1}) \sum_{\alpha \prec \beta} \mathrm{sign}(\alpha,\beta) e_{-\alpha} \wedge_{-\mathrm{sign}(\alpha,\beta)\epsilon_{\alpha,\beta}} e_\beta \qquad (3.4)$$

Combining (3.4) with Table 4.2, we obtain (3.3). This proves that $\bar{R}_{\mathrm{GGS}} = \bar{R}_J$ and hence that $R_{\mathrm{GGS}} = R_J$ in the case $\Gamma_1 \cap \Gamma_2 = \emptyset$. The proof is finished. $\square$

3.2 The generalized disjoint case

In fact, the results we have obtained extend easily to the *generalized disjoint* case:

Definition 3.1 *A triple* $(\Gamma_1, \Gamma_2, \tau)$ *is said to be generalized disjoint if* $\Gamma_1 = \bigcup_{i=1}^m \Gamma_1^i$ *where* $\Gamma_1^i \perp \Gamma_1^j, i \neq j$ *and* $\tau(\Gamma_1^i) \cap \Gamma_1 \subset \Gamma_1^{i+1}, i < m$, *while* $\tau\Gamma_1^m \subset \Gamma_2$.

Theorem 3.2 *For any generalized disjoint triple,* $R_J = R_{GGS}$.

Proof. Note first by (1.11), (1.5) that $K_{\alpha,\beta}$ and $\epsilon_{\alpha,\beta}$ are as given in Tables 4.1, 4.2, respectively. As in the disjoint case, we have the following main property:

Lemma 3.3 *If* $(e_{-\alpha} \otimes e_{\tau^x\alpha})(e_{-\beta} \otimes e_{\tau^y\beta})_{ab} \neq 0$, *for* $\{a,b\} = \{1,2\}$, *then* $(a,b) = (1,2)$, $x = y$, *and* τ^x *reverses orientation on* $\alpha + \beta$.

Proof. Suppose $(a,b) = (2,1)$. Then $\{\tau^x\alpha, \beta\} \subset \tilde{\Gamma}_1^i$ for some i. But then, $\alpha \in \Gamma_1^j, \tau^y\beta \in \Gamma_1^k$ for $j < i < k$, so $e_{-\alpha} e_{\tau^y\beta} = 0$, a contradiction. So, $(a,b) = (1,2)$. Now, this implies that $\{\alpha,\beta\} \subset \Gamma_1^i, \{\tau^x\alpha, \tau^y\alpha\} \subset \Gamma_1^j$ for some $i < j$. Then $x = y = j - i$. $\square$

Because of this fact, we may set J^{ij} to be the J-matrix corresponding to the disjoint triple $(\Gamma_1^i, \tau\Gamma_1^{j-1}, \tau^{j-i})$, and similarly define $\bar{R}_J^{ij}, R_{\mathrm{GGS}}^{ij}$, and $A^{ij} = J^{ij} - 1$, so that $A^{ij} X A^{kl} = 0$ whenever $(i,j) \neq (k,l)$. Thus, $J = \prod_{i<j} J^{ij} = 1 + \sum_{i<j} A^{ij}$, and $A^{ij} X A^{kl} = 0$ whenever $i \neq j$. Hence, $\bar{R}_J = R_s + \sum_{i,j} \bar{(R_J^{ij}} - R_s) = R_s + \sum_{i,j} (\bar{R}_{\mathrm{GGS}}^{ij} - R_s) = \bar{R}_{\mathrm{GGS}}$. $\square$

In the appendix we prove the twist conjecture and hence the GGS conjecture in the orthogonal generalized disjoint case (defined in Section 1.4) by demonstrating that R_J satisfies the QYBE.

4 The Cremmer-Gervais triple

In this section we prove $R_J = R_{\mathrm{GGS}}$ in the Cremmer-Gervais case, and hence the twist conjecture since the GGS conjecture is proved in this case (for example, see [8].)

Theorem 4.1 *For the Cremmer-Gervais triple* $(\{\alpha_1, \ldots, \alpha_{n-1}\}, \{\alpha_2, \ldots, \alpha_n\}, \tau)$, $\tau\alpha_i = \alpha_{i+1}$, *one has* $R_J = R_{GGS}$.

Proof. Note first from (1.11), (1.5) that $K_{\alpha,\beta} = \frac{1}{2}[\alpha \lessdot \beta] + [\alpha \ll \beta]$, and $\epsilon_{\alpha,\beta} = -K_{\alpha,\beta}$. Next, note that $(e_{-\alpha} \otimes e_{\tau^x\alpha})(e_{-\beta} \otimes e_{\tau^y\beta})_{ab} \neq 0$ for $\{a,b\} = \{1,2\}$ iff $(a,b) = (1,2)$ and $y = x + |\alpha| + |\beta|$. Hence, considering the product form (1.10), we find that $J^{-1} = 1 + \sum_{\alpha \prec \beta} q^{K_{\alpha,\beta}}(q^{-1} - q)e_\beta \otimes e_{-\alpha}$. Furthermore, if we set $B = J^{-1} - 1, A = J - 1$, we see that

$$\bar{R}_J = J^{-1}R_sJ_{21} = R_s + B + A_{21} + (q-1)A_{21}^{\lessdot} + (q - q^{-1})PA_{21}^{\ll} + (q-q^{-1})PB_{21} + (q-q^{-1})PB_{21}A_{21} + BA_{21}. \quad (4.1)$$

Since, in addition, $(e_\beta \otimes e_{-\alpha})_{ab} \in G$ for $\alpha, \beta > 0, \{a,b\} = \{1,2\}$ iff $\alpha \prec \beta$, we may infer that $(\bar{R}_J)_+ = (\bar{R}_{\mathrm{GGS}})_+$. Thus, it suffices to show $(\bar{R}_J)_{-\beta,-\alpha} = (\bar{R}_{\mathrm{GGS}})_{-\beta,-\alpha}$ whenever $\alpha \prec \beta$ (equivalently, $\alpha < \beta$.) We proceed inductively on the index under τ. Suppose this is true for all $\alpha' \prec \beta'$ where $i_\tau(\beta' - \alpha') < i_\tau(\beta - \alpha)$. Since, with respect to the ordering in (1.10), $(e_{-\alpha} \otimes e_\beta)(e_{-\alpha'} \otimes e_{\beta'}) \neq 0$ only when $(\alpha, \beta) < (\alpha', \beta')$, we may rewrite (4.1) using our inductive hypothesis as

$$(\bar{R}_J)_{-\beta,-\alpha} = (q - q^{-1})\big[q^{K_{\alpha,\beta}+[\alpha\lessdot\beta]} + [\alpha \overline{<} \beta]q + (PB_{21})_{-\beta,-\alpha} + \big(\bar{R}_{\mathrm{GGS}} \sum_{\alpha \prec \beta} q^{K_{\alpha,\beta}}(e_{-\alpha} \otimes e_\beta)\big)_{-\beta,-\alpha}\big]. \quad (4.2)$$

When we write out the sum above, we will add it in pairs corresponding to the bijections f, g defined in Section 3; this will make the sum cancel. We separately consider the cases $\alpha \overline{<} \beta, \alpha \lessdot \beta, \alpha \ll \beta$.

First suppose $\alpha \overline{<} \beta$. Set $\alpha = e_j - e_i, \beta = e_k - e_{i+k-j}$ where $j < k < i < i+k-j$. Then $(R_{\mathrm{GGS}})_{-\beta,-\alpha} = q - q^{-1}$. By (4.2), we may write

$$(\bar{R}_J)_{-\beta,-\alpha} = (q-q^{-1})\big[1 + q - q + \sum_{l=j+1}^{k} \big((q-q^{-1})(e_{il} \otimes e_{k,i+k-l})q(e_{lj} \otimes e_{i+k-l,i+k-j}) - q(q-q^{-1})(e_{i,i+l-j} \otimes e_{k,k-l+j})q^0(e_{i+l-j,j} \otimes e_{k-l+j,i+k-j})\big)_{-\beta,-\alpha}\big] = q - q^{-1}. \quad (4.3)$$

Now, suppose $\alpha \lessdot \beta$. In this case, set $\alpha = e_j - e_i, \beta = e_i - e_{2i-j}$. Now, (4.2) becomes

$$(\bar{R}_J)_{-\beta,-\alpha} = (q-q^{-1})\big[q^{3/2} - q^{1/2}(q-q^{-1}) + \sum_{l=i+1}^{j} \big(q^{-1/2}(q-q^{-1})(e_{il} \otimes e_{i,2i-l})q(e_{lj} \otimes e_{2i-l,2i-j}) - q^{1/2}(q-q^{-1})(e_{i,2i-l} \otimes e_{i,l})q^0(e_{2i-l,j} \otimes e_{l,2i-j})\big)_{-\beta,-\alpha}\big] = q^{-1/2}(q-q^{-1}). \quad (4.4)$$

Finally, suppose $\alpha \ll \beta$. In this case, $\alpha = e_j - e_i, \beta = e_k - e_{i+k-j}$ where $j < i < k < i + k - j$, and we write

$$(\bar{R}_J)_{-\beta,-\alpha} = (q-q^{-1})\big[q-(q-q^{-1}) + \sum_{l=j+1}^{k}\big(q^{-1}(q-q^{-1})(e_{il}\otimes e_{k,i+k-l})q(e_{lj}\otimes e_{i+k-l,i+k-j}) - (q-q^{-1})(e_{i,i+l-j}\otimes e_{k,k-l+j})q^0(e_{i+l-j,j}\otimes e_{k-l+j,i+k-j})\big)_{-\beta,-\alpha}\big] = q^{-1}(q-q^{-1}). \quad (4.5)$$

The proof is finished. □

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A Proof that R_J satisfies the QYBE in special cases

A.1 The disjoint case, by P.Etingof and T.Schedler

In this subsection we will prove Theorem 1.4.i by showing that R_J satisfies the QYBE for disjoint triples (i.e. $\Gamma_1 \cap \Gamma_2 = \emptyset$). We note that in the case when Γ_1 is orthogonal to Γ_2, this was done (using the same method) by T.Hodges. This is sufficient given Theorem 3.1.

Let $U_\hbar(\mathfrak{sl}(n))$ be the quantum universal enveloping algebra generated by $E_{\alpha_i}, F_{\alpha_i}, H_{\alpha_i}$, $\alpha_i \in \Gamma$, under the relations

$$[H_{\alpha_i}, E_{\alpha_j}] = (\alpha_i,\alpha_j)E_{\alpha_j}, [H_{\alpha_i}, F_{\alpha_j}] = -(\alpha_i,\alpha_j)F_i, [H_{\alpha_i}, H_{\alpha_j}] = 0, \quad (A.1)$$

$$[E_{\alpha_i}, F_{\alpha_j}] = \delta_{ij}\frac{q^{H_{\alpha_i}} - q^{-H_{\alpha_i}}}{q-q^{-1}}, \quad (A.2)$$

$$E_{\alpha_i}^2 E_{\alpha_{i\pm1}} - (q+q^{-1})E_{\alpha_i}E_{\alpha_{i\pm1}}E_{\alpha_i} + E_{\alpha_{i\pm1}}E_{\alpha_i}^2 = 0, \quad (A.3)$$

$$F_{\alpha_i}^2 F_{\alpha_{i\pm1}} - (q+q^{-1})F_{\alpha_i}F_{\alpha_{i\pm1}}F_{\alpha_i} + F_{\alpha_{i\pm1}}F_{\alpha_i}^2 = 0, \quad (A.4)$$

where the coproduct, counit, and antipode are given by

$$\Delta(E_{\alpha_i}) = E_{\alpha_i}\otimes q^{H_{\alpha_i}} + 1\otimes E_{\alpha_i}, \Delta(F_{\alpha_i}) = F_{\alpha_i}\otimes 1 + q^{H_{\alpha_i}}\otimes F_{\alpha_i}, \quad (A.5)$$

$$\Delta(H_{\alpha_i}) = H_{\alpha_i}\otimes 1 + 1\otimes H_{\alpha_i}, \epsilon(F_{\alpha_i}) = \epsilon(E_{\alpha_i}) = \epsilon(H_{\alpha_i}) = 0, \quad (A.6)$$

$$S(E_{\alpha_i}) = -E_{\alpha_i}q^{H_{\alpha_i}}, S(F_{\alpha_i}) = -q^{H_{\alpha_i}}F_{\alpha_i}, S(H_{\alpha_i}) = -H_{\alpha_i}. \quad (A.7)$$

We will use the representation $\phi : U_\hbar(\mathfrak{sl}(n)) \to Mat_n(\mathbb{C})$ by $\phi(E_{\alpha_i}) = e_{i,i+1}, \phi(F_{\alpha_i}) = e_{i+1,i}$, and $\phi(H_{\alpha_i}) = e_{ii} - e_{i+1,i+1}$.

Now, we recall the results of Hodges [7] using the notation of [3]. Fix a disjoint Belavin-Drinfeld triple $(\Gamma_1, \Gamma_2, \tau)$. Let $\mathfrak{h}_i$ be the subpaces of $\mathfrak{h}$ spanned by $e_{\alpha_k}, \alpha_k \in \Gamma_i$. Let U_i be the Hopf subalgebras of $U_\hbar(\mathfrak{sl}(n))$ generated by $E_{\alpha_k}, F_{\alpha_k}, H_{\alpha_k}, \alpha_k \in \Gamma_i$. Define $f_\tau : U_1 \to U_2$

by $f_\tau(E_{\alpha_i}) = E_{\tau\alpha_i}, f_\tau F_{\alpha_i} = F_{\tau\alpha_i}, f_\tau H_{\alpha_i} = H_{\tau\alpha_i}$. It is clear that f_τ is an isomorphism of Hopf subalgebras. Let $g_\tau : \phi(U_1) \to \phi(U_2)$ be the homomorphism descending from f_τ.

Define $Z = (g_\tau \otimes 1)\Omega_{\mathfrak{h}_1}$ where $\Omega_{\mathfrak{t}}$ denotes the Casimir element of the usual bilinear form on a nondegenerate subspace $\mathfrak{t} \subset \mathfrak{h}$. Let $T \in \mathfrak{h} \otimes \mathfrak{h}$ be a solution of the following equations:

$$(x \otimes 1, T) = (1 \otimes \tau(x), T) = 0, \quad (\tau(x) \otimes 1 + 1 \otimes x, Z - T) = 0. \tag{A.8}$$

for any $x \in \mathfrak{h}_1$. Define

$$\mathcal{J} = q^T \mathcal{J}_0, \quad \mathcal{J}_0 = 1 + \sum_{\alpha \prec \beta} (q - q^{-1})(-q^{-1})^{C_\alpha(|\alpha|-1)} e_\beta \otimes e_{-\alpha}, \tag{A.9}$$

where $C_\alpha = 1$ if $\alpha \prec^{\leftarrow} \beta$ and 0 if $\alpha \prec^{\rightarrow} \beta$.

The following proposition can be deduced from the results of [7].

Proposition A.1 *The element $R_{\mathcal{J}} \equiv \mathcal{J}^{-1} R_s \mathcal{J}_{21} \in Mat_n(\mathbb{C}) \otimes Mat_n(\mathbb{C})$ satisfies the Hecke relation and the quantum Yang-Baxter equation.*

Proof. Define

$$\mathcal{J}' = q^{T-Z}(\tau \otimes 1)(\phi \otimes \phi)(\mathbb{R}) \tag{A.10}$$

where $\mathbb{R}$ is the universal R-matrix of the Hopf subalgebra U_1. By Proposition 4.1 of [3], $(\mathcal{J}')^{-1} R_s (\mathcal{J}')_{21}$ satisfies the Hecke relation and the quantum Yang-Baxter equation, so it is enough to show that $\mathcal{J}$ coincides with $\mathcal{J}'$. This can be deduced from an explicit formula for the universal R-matrix $\mathbb{R}$. We use the formula given in [9] and evaluate it in the representation as follows:

$$(\phi \otimes \phi)\big((f_\tau \otimes 1)(\mathbb{R})\big) = \big(1 + \sum_{\alpha \prec \beta} (q - q^{-1})(-q^{-1})^{[\alpha \prec^{\leftarrow} \beta](|\alpha|-1)} e_\beta \otimes e_{-\alpha}\big) q^Z. \tag{A.11}$$

Here we take our normal ordering to be given by left to right on the Dynkin diagram (for simple roots). The additional powers of $-q^{-1}$ in the reversing case appear because in [9], $E_{\alpha+\beta} = E_\alpha E_\beta - q^{(\alpha,\beta)} E_\beta E_\alpha$, and so $\phi(E_{\tau\alpha}) = (-q^{-1})^{|\alpha|-1} e_{\tau\alpha}$ when τ reverses order on α, while $\phi(E_{\tau\alpha}) = e_{\tau\alpha}$ when τ preserves order on α.

All that remains is to show $[q^Z, (\phi \otimes \phi)(\mathcal{J}_0)] = 0$. Observe that $\sum_i e_{ii} \otimes e_{ii} = \Omega_h = \Omega_{h_1} + \Omega_{h_1^\perp}$ so that $[\Omega_{h_1}, (\phi \otimes \phi)(\mathbb{R})] = 0$. This finishes the proof. $\square$

We denote the usual inner product on $(\mathfrak{h}_1 \oplus \mathfrak{h}_2)$ by $I(x, y)$, so that we may define the bilinear form $B(x, y) \equiv (x, \tau y)$ defined on $(\mathfrak{h}_1 \oplus \mathfrak{h}_2) \otimes \mathfrak{h}_1$. Similarly define $B^T(x, y) = (\tau x, y)$ on $\mathfrak{h}_1 \otimes (\mathfrak{h}_1 \oplus \mathfrak{h}_2)$. Since $\mathfrak{h}_1 + \mathfrak{h}_2$ has a nondegenerate inner product, any element of $(\mathfrak{h}_1 \oplus \mathfrak{h}_2)^{\otimes 2}$ can be regarded as a bilinear form on $\mathfrak{h}_1 \oplus \mathfrak{h}_2$, by $X \mapsto F_X(a, b) = (a \otimes b, X)$. Thus such an element can be written as a 2 by 2 matrix whose ij-th entry is a form on $\mathfrak{h}_i \otimes \mathfrak{h}_j$. We will use such notation below.

Lemma A.1 *There exists a unique solution T of the equations* (A.8) *in $(\mathfrak{h}_1 \oplus \mathfrak{h}_2)^{\otimes 2}$. This solution has the form*

$$T = \begin{pmatrix} 0 & 0 \\ I + B & 0 \end{pmatrix}. \tag{A.12}$$

Proof. The proof is by a direct computation. $\square$

Now let us compute $R_{\mathcal{J}}$. We have

$$R_{\mathcal{J}} = \mathcal{J}_0^{-1} q^{-T} R_s q^{T_{21}} (\mathcal{J}_0)_{21}. \tag{A.13}$$

Using the fact that

$$q^X R_s q^{-X} = R_s, X \in S^2\mathfrak{h} \tag{A.14}$$

one transforms (A.13) to

$$R_{\mathcal{J}} = \mathcal{J}_0^{-1} q^{T_{21}} R_s q^{-T} (\mathcal{J}_0)_{21}. \tag{A.15}$$

It is clear that T_{21} commutes with $\mathcal{J}_0$, since both components commute separately, thus

$$R_{\mathcal{J}} = q^{T_{21}} \mathcal{J}_0^{-1} R_s (\mathcal{J}_0)_{21} q^{-T}. \tag{A.16}$$

Let J be as defined in (1.7) and (1.8), using Table 1. Then $J = q^Y \mathcal{J}_0 q^{-Y}$, where $Y = \frac{1}{2}\sum e_{ii} \otimes e_{ii}$. Thus, using (A.14) again, we get

$$R_{\mathcal{J}} = q^{T_{21}-Y} J^{-1} R_s J_{21} q^{-T+Y}. \tag{A.17}$$

Setting $R_J = q^{r^0} J^{-1} R_s J_{21} q^{r^0}$, we get

$$R_{\mathcal{J}} = q^{T_{21}-Y-r^0} R_J q^{-T+Y-r^0}, \tag{A.18}$$

Write Y as $Y_0 + Y'$, where $Y_0 \in (\mathfrak{h}_1 \oplus \mathfrak{h}_2)^{\otimes 2}$, $Y' \in ((\mathfrak{h}_1 \oplus \mathfrak{h}_2)^{\perp})^{\otimes 2}$. It is clear that $q^{Y'}$ commutes with R_J, so we obtain

$$R_{\mathcal{J}} = q^{T_{21}-Y_0-r^0} R_J q^{-T+Y_0-r^0}. \tag{A.19}$$

It is easy to see that $Y_0 = \frac{1}{2}\begin{pmatrix} I & I \\ I & I \end{pmatrix}$, and r^0 can be chosen to be the element of $(\mathfrak{h}_1 \oplus \mathfrak{h}_2)^{\otimes 2}$ given by the matrix $\begin{pmatrix} 0 & \frac{1}{2}(I + B^T) \\ -\frac{1}{2}(I+B) & 0 \end{pmatrix}$. Therefore, $R_{\mathcal{J}} = q^{-U} R_J q^U$, where $U = \begin{pmatrix} I/2 & -B^T/2 \\ -B/2 & I/2 \end{pmatrix}$.

Let $W = \begin{pmatrix} I/2 & B^T/2 \\ B/2 & I/2 \end{pmatrix}$. Then $W \in S^2 K$, where K is the Lie algebra of symmetries of R_J, i.e. $K = \{x \in Mat_n(\mathbb{C}) \mid [1 \otimes x + x \otimes 1, R_J] = 0\}$. Therefore, $q^W R_J q^{-W} = (D \otimes D) R_J (D^{-1} \otimes D^{-1})$ for a suitable diagonal matrix D. So in order to finish the proof of the twist conjecture, it suffices to show that $q^{U+W} R_{\mathcal{J}} q^{-U-W}$ is a solution of the Yang-Baxter equation.

Note that $U + W = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$ and therefore $U + W \in S^2(\mathfrak{h}_2^{\perp}) \oplus S^2(\mathfrak{h}_1^{\perp})$. Thus, the twist conjecture follows from the following lemma.

Lemma A.2 *Let X be an element of $S^2(\mathfrak{h}_1 \oplus \mathfrak{h}_2)$ which is orthogonal to $\mathfrak{h}_1 \otimes \mathfrak{h}_2$. Then $q^X R_J q^{-X} = (D \otimes D)J(D^{-1} \otimes D^{-1})$ for a suitable diagonal matrix D.*

Proof. We can write X as $X_1 + X_2$, where X_1 is orthogonal to $\mathfrak{h}_1$ in both components and X_2 to $\mathfrak{h}_2$. It suffices to prove the lemma for each of them separately. Let us do it for X_1, for X_2 the proof is analogous.

Let $E_\alpha = e_{\tau\alpha} \otimes e_{-\alpha}$. It is clear that $q^{X_1} E_\alpha q^{-X_1} = \lambda(\alpha) E_\alpha$ for some eigenvalue $\lambda(\alpha)$. All we need to show is that $\lambda(\alpha + \beta)$, when it is defined, equals $\lambda(\alpha)\lambda(\beta)$.

We may assume that $\alpha \lessdot \beta$, i.e. α is to the left of β. If τ reverses orientation on $\alpha + \beta$, i.e. $\alpha + \beta \prec^{\leftarrow} \tau(\alpha + \beta)$, then the statement is clear since $E_{\alpha+\beta} = E_\beta E_\alpha$. If not, we have $E_{\alpha+\beta}^{t_2} = E_\alpha^{t_2} E_\beta^{t_2}$ (t_2 is transposition in the second component). Thus, using that the second component of X_1 commutes with e_α, e_β, we get

$$(q^{X_1} E_{\alpha+\beta} q^{-X_1})^{t_2} = q^{X_1} E_\alpha^{t_2} E_\beta^{t_2} q^{-X_1} = \lambda(\alpha)\lambda(\beta) E_\alpha^{t_2} E_\beta^{t_2} = \lambda(\alpha)\lambda(\beta) E_{\alpha+\beta}^{t_2}. \tag{A.20}$$

This implies the lemma. $\square$

A.2 The orthogonal generalized disjoint case

In this subsection we prove Theorem 1.4.ii by showing R_J satisfies the QYBE in the orthogonal generalized disjoint case, defined in Definition 1.3. This is sufficient due to Theorem 3.2.

Set $\mathfrak{h}_{kl} = \mathrm{Span}(H_{\alpha_i} \mid \alpha_i \in \Gamma_1^k, \tau^{l-1}\alpha_i \in \Gamma_1)$. We also consider U_{kl}, the Hopf subalgebra of $U_\hbar(\mathfrak{sl}(n))$ generated by $E_{\alpha_i}, F_{\alpha_i}, H_{\alpha_i}$, for $\alpha_i \in \mathfrak{h}_{kl}$, and U'_{kl}, the Hopf subalgebra of $U_\hbar(\mathfrak{sl}(n))$ generated by $E_{\tau^l\alpha_i}, F_{\tau^l\alpha_i}, H_{\tau^l\alpha_i}$. Define the map $f_{\tau^l} : U_{kl} \to U'_{kl}$ by $f_{\tau^l}(E_{\alpha_i}) = E_{\tau^l\alpha_i}, f_{\tau^l} F_{\alpha_i} = F_{\tau^l\alpha_i}, f_{\tau^l} H_{\alpha_i} = H_{\tau^l\alpha_i}$. Then it is easy to check that f_{τ^l} is a Hopf algebra isomorphism. Also, define the map $g_{\tau^l} : \phi(U_{kl}) \to \phi(U'_{kl})$ to be the homorphism descending from f_{τ^l}.

Now, define $\mathbb{R}^{kl} = (f_{\tau^l} \otimes 1)(\mathbb{R}')^{kl}$ where $(\mathbb{R}')^{kl}$ is the universal R-matrix of U_{kl}. Now, we define the twist $\mathcal{J}$ by $\mathcal{J} = \prod_{i=1}^m \prod_{j=1}^{m+1-i} \mathbb{R}^{j,j+i}$. Recall from [3] the definition of a twist:

Definition A.1 *$\mathcal{J} \in U_q(\mathfrak{sl}(n))$ is a twist if $\mathcal{J} \equiv 1 \pmod{\hbar}, (\epsilon \otimes 1)\mathcal{J} = (1 \otimes \epsilon)\mathcal{J} = 1$, and $(\Delta \otimes 1)(\mathcal{J})\mathcal{J}_{12} = (1 \otimes \Delta)(\mathcal{J})\mathcal{J}_{23}$. In addition, a twist is triangular if $\mathcal{J} \in U_{\geq 0} \otimes U_{\leq 0}$, where $U_{\geq 0}, U_{\leq 0} \subset U_\hbar(\mathfrak{sl}(n))$ are the Hopf subalgebras generated by the $E_{\alpha_i}, H_{\alpha_i}$, and by the $F_{\alpha_i}, H_{\alpha_i}$, respectively.*

Theorem A.1 *$\mathcal{J}$ is a triangular twist.*

Proof. It is obvious that $\mathcal{J} \equiv 1 \pmod{\hbar}$, $(\epsilon \otimes 1)\mathcal{J} = (1 \otimes \epsilon)\mathcal{J} = 1$, and that $\mathcal{J}$ is triangular. It thus suffices to prove $(\Delta \otimes 1)(\mathcal{J})\mathcal{J}_{23} = (1 \otimes \Delta)(\mathcal{J})\mathcal{J}_{12}$. By construction, one easily sees that $[\mathbb{R}_{ab}^{kl}, \mathbb{R}_{a'b'}^{k'l'}] = 0$ for $1 \leq a < b \leq 3, 1 \leq k < l \leq m+1$ and similarly for a', b', k', l' in the event that $\{(a,k),(b,l)\} \cap \{(a',k'),(b',l')\} = \emptyset$. Furthermore, since $\mathbb{R}$ satisfies the QYBE, where $\mathbb{R}$ is the universal R-matrix of any Hopf subalgebra of $U_h(\mathfrak{sl}(n))$, we find $\mathbb{R}_{12}^{ij}\mathbb{R}_{13}^{ik}\mathbb{R}_{23}^{jk} = \mathbb{R}_{23}^{jk}\mathbb{R}_{13}^{ik}\mathbb{R}_{12}^{ij}$ for $i < j < k$. Thus, the theorem follows from the following combinatorial lemma:

Lemma A.3 *Let G be a semigroup generated by the set $T = \{r_{ij}^{ab}, 1 \leq i < j \leq 3, 1 \leq a < b \leq n\}$ for a given $n \in Z^+$. Consider the relations*

$$r_{ij}^{ab} r_{kl}^{cd} = r_{kl}^{cd} r_{ij}^{ab}, \quad \{(i,a),(j,b)\} \cap \{(k,c),(l,d)\} = \emptyset \tag{A.21}$$

$$r_{12}^{ab} r_{13}^{ac} r_{23}^{bc} = r_{23}^{bc} r_{13}^{ac} r_{12}^{ab}, \quad 1 \leq a < b < c \leq n. \tag{A.22}$$

Then, if G satisfies relations (A.21), (A.22), *it also satisfies the relation*

$$\left(\prod_{i=1}^{n-1}\prod_{j=1}^{n-i} r_{13}^{j,i+j} r_{12}^{j,i+j}\right) \prod_{i=1}^{n-1}\prod_{j=1}^{n-i} r_{23}^{j,i+j} = \left(\prod_{i=1}^{n-1}\prod_{j=1}^{n-i} r_{13}^{j,i+j} r_{23}^{j,i+j}\right) \prod_{i=1}^{n-1}\prod_{j=1}^{n-i} r_{12}^{j,i+j}. \tag{A.23}$$

Proof. Let F be the free semigroup generated by the set T as above, and let Y be the set of relations (A.21),(A.22). Note that in each side of (A.23), every generator of G appears exactly once. Let $H \subset F$ be the set of such elements of F, so that for any element $X \in H$, we can say $r_1 <_X r_2$ for generators r_1, r_2 if r_1 appears to the left of r_2 in X. Let L, R denote the left and right hand sides of (A.23), respectively, considered as elements of H. Note that both L and R satisfy

$$r_{13}^{ab} <_X r_{12}^{ac}, r_{13}^{ef} <_X r_{23}^{df}, \quad b \leq c, d \leq e \tag{A.24}$$

when one replaces X with L, R. Again replacing X with L, R we find that

$$r_{12}^{ab} <_X r_{13}^{ac} <_X r_{23}^{bc} \quad \text{or} \quad r_{23}^{bc} <_X r_{13}^{ac} <_X r_{12}^{ab}, \quad 1 \leq a < b < c \leq n. \tag{A.25}$$

Finally, consider the property

$$r_{ij}^{ab} <_X r_{ij}^{cd} \Leftrightarrow r_{ij}^{ab} <_L r_{ij}^{cd}, \text{whenever } a = c \text{ or } b = d. \tag{A.26}$$

Obviously L satisfies (A.26), and it is easy to see that R does as well. Let $K \subset H$ be the set of elements of H satisfying (A.24), (A.25), (A.26). Define a function $f : K \to \mathbb{Z}^+$ by $f(X) = |\{(a,b,c) \in \mathbb{Z}^3 | 1 \leq a < b < c \leq n, r_{12}^{ab} <_X r_{13}^{ac} <_X r_{23}^{bc}\}|$. We claim that $|K/Y| = 1$, that is, the image of K is just one element in the natural map $F \to F/Y = G$. To show this, first note that if $X \in K$ satisfies $f(X) = 0$, then for all pairs of generators $r_1, r_2 \in T$ that do not appear in (A.21), $r_1 <_X r_2 \Leftrightarrow r_1 <_L r_2$. This follows from (A.24), (A.25), and $f(X) = 0$. Thus, $f(X) = 0 \Rightarrow X \equiv L \pmod{Y}$. Now, we show that given $f(X) = m$, $\exists Z \in H$ such that $X \equiv Z \pmod{Y}$ and $f(Z) = m - 1$, whenever $m \in \mathbb{Z}^+$. Indeed, take a triple (a,b,c) so that $r_{23}^{bc} <_X r_{13}^{ac} <_X r_{12}^{ab}$, and so that for any other triple (a',b',c') satisfying this property, $b' - a' \geq b - a$, and if $a = a', b = b'$, then $c' < c$. Clearly there exists such a triple.

Now, consider terms $r \in T$ with $r_{23}^{bc} <_X r <_X r_{13}^{ac}$. We will consider all cases when r does not commute with r_{23}^{bc} or r_{13}^{ac} under (A.21). There are two possibilities: (i) $r = r_{23}^{dc}$. Now, (A.26) implies $d < b$, and (A.24) implies $a < d$. But this contradicts our choice of a, b, c. (ii) $r = r_{13}^{dc}$. Here (A.26) implies $d > a$, and (A.24) implies $d < b$. This contradicts our choice of a, b, c. Hence, $X \equiv X' \pmod{Y}$ where r_{23}^{bc} appears next to r_{13}^{ac}, and $f(X') = m$.

Now, consider terms $r \in T$ with $r_{13}^{ac} <_{X'} r <_{X'} r_{12}^{ab}$ that don't commute with r_{12}^{ab}. Contradictions will be made with our choice of a, b, c. There are four possibilities. (i) $r = r_{12}^{ad}$: (A.26) implies $d < b$, contradiction. (ii) $r = r_{12}^{db}$: (A.26) implies $a < d$, contradiction. (iii)

$r = r^{ad}_{13}$: (A.26) implies $c < d$, contradiction. (iv) $r = r^{bd}_{23}$: (A.26) implies $c < d$, contradiction. Hence, $X' \equiv X''$ (mod Y) where the terms $r^{bc}_{23}, r^{ac}_{13}, r^{ab}_{12}$ appear consecutively, and $X'' \equiv Z$ (mod Y) where Z differs from X'' by swapping r^{bc}_{23}, r^{ab}_{12}. Since $f(Z) = m - 1$, the proof is finished. □

Corollary A.1 *The element $\mathcal{J}_{21}^{-1}\mathcal{R}\mathcal{J}$ satisfies the QYBE, where $\mathcal{R}$ is the universal R-matrix for $U_h(\mathfrak{sl}(n))$. Hence, $(\phi \otimes \phi)(\mathcal{J})^{-1} R_s (\phi \otimes \phi)(\mathcal{J})_{21}$ satisfies the QYBE.*

Proof. This is clear since $q^{-\frac{1}{n}}(R_s)_{21} = (\phi \otimes \phi)(\mathcal{R})$ (which is an easy consequence of the formula in [9]). □

Proposition A.2 *For a suitable r^0 and some $x \in S^2\mathfrak{h}$, $Jq^{-r^0} = q^x(\phi \otimes \phi)(\mathcal{J})$.*

Proof. First, we explicitly evaluate $(\phi \otimes \phi)(\mathbb{R}^{kl})$ using the formulas in [9]. Indeed, we may write

$$(\phi \otimes \phi)(\mathbb{R}^{kl}) = \sum_{\alpha \in \tilde{\Gamma}_k \cap \tau^{-l}\tilde{\Gamma}} (-q^{-1})^{[\alpha \prec \leftarrow \beta](|\alpha|-1)} e_{\tau^l \alpha} \otimes e_{-\alpha} q^{Z^{kl}}, \tag{A.27}$$

where $Z^{kl} = (g_{\tau^l} \otimes 1)\Omega_{\mathfrak{h}_{kl}}$ where $\Omega_{\mathfrak{h}_{kl}}$ is the Casimir element for the space $\mathfrak{h}_{kl}$ with the usual bilinear form. The coefficients $(-q^{-1})^{[\alpha \prec \leftarrow \beta](|\alpha|-1)}$ follow easily from the fact that $E_{\alpha+\beta} = E_\alpha E_\beta - q^{(\alpha,\beta)} E_\beta E_\alpha$ (as defined in [9]) upon evaluation in $Mat_n(\mathbb{C}) \otimes Mat_n(\mathbb{C})$, since different terms vanish in the reversing and non-reversing cases. Note that $(\phi \otimes \phi)(\mathbb{R}^{kl})_{\alpha, \tau^l \alpha} = J_{\alpha, \tau^\alpha}$ for $\alpha \in \tilde{\Gamma}_k \cap \tau^{-l}\tilde{\Gamma}$. It remains to reconcile the extra terms, q^{-r^0} and the $q^{Z^{kl}}$.

As in the previous subsection, one sees that $\Omega_{\mathfrak{h}} = \Omega_{\mathfrak{h}_{kl}^{\perp}} + \Omega_{\mathfrak{h}_{kl}}$ where $\mathfrak{h}_{kl}^{\perp}$ is the orthogonal complement to $\mathfrak{h}_{kl}$ in $\mathfrak{h}$. Since $\Omega_{\mathfrak{h}} = \sum_i e_{ii} \otimes e_{ii}$, we have $[\Omega_{\mathfrak{h}}, (\phi \otimes \phi)(\mathbb{R}')^{kl}] = 0$, and it follows that $[q^{Z^{kl}}, (\phi \otimes \phi)(\mathbb{R}^{kl})] = 0$. By orthogonality, $[q^{Z^{kl}}, (\phi \otimes \phi)(\mathbb{R}^{k'l'})] = 0$ for $(k', l') \neq (k, l)$. Now, set $Y = \sum_{k,l} Z^{kl}$. Then, $\mathcal{J} = Jq^Y$. Now, it is clear that $(\alpha \otimes \beta, Y) = (\alpha, \tau^l \beta)$ for $\alpha \in \Gamma_1^j \cup \tau\Gamma_1^{j-1}, \beta \in \Gamma_1^{j+l} \cup \tau\Gamma_1^{j+l-1}$ where we again take $\Gamma_1^0 = \Gamma_1^{m+1} = \emptyset$. Then, we may take $r^0 = \frac{1}{2}(Y_{21} - Y)$, so that $r^0 + Y \in S^2\mathfrak{h}$. Furthermore, as we have seen, $[q^Y, \mathcal{J}] = 0$, and it is clear that $[q^{Y_{21}}, \mathcal{J}] = 0$ by orthogonality. The proof is finished. □

Corollary A.2 *R_J satisfies the QYBE. Theorem 1.4 is proved.*

Proof. Clear. □

B Proof of Giaquinto's formula (1.6) for R_{GGS} in the generalized Cremmer-Gervais case

In this section we explicitly compute R_{GGS} for generalized Cremmer-Gervais triples, the only triples satisfying $|\Gamma_1| + 1 = |\Gamma|$ (omitting only one root). These are precisely the cases where r^0 is unique if its first component has trace zero ($r^0 \in \wedge^2\mathfrak{h}'$ where $\mathfrak{h}' \subset \mathfrak{h}$ is the subspace of diagonal matrices of trace zero.) First we summarize the results given in Example 1.1 as proved in [4]. Let Res give the residue mod n in $\{1, \ldots, n\}$. Take the triple indexed by (n, m), where $\tau\alpha_i = \alpha_{\text{Res}(i+m)}$. The unique r^0 whose first component has trace zero is given

by $(r^0)^{ii}_{ii} = 0, (r^0)^{ij}_{ij} = \frac{1}{2} - \mathrm{Res}(\frac{j-i}{m})$. Then, the only difficulty is in computing $q^{r^0}\tilde{a}q^{r^0}$, so here we use (1.5) to prove $q^{r^0}\tilde{a}q^{r^0} = \sum_{\alpha \prec \beta} e_{-\alpha} \wedge_{\frac{-2O(\alpha,\beta)}{n}} e_\beta$.

Clearly we have

$$q^{r^0}\tilde{a}q^{r^0} = \sum_{\alpha \prec \beta} e_{-\alpha} \wedge_{\epsilon_{\alpha,\beta}+r(\alpha,\beta)} e_\beta, \tag{B.1}$$

where $r(e_j - e_i, e_k - e_{i+k-j}) = (r^0)^{j,i+k-j}_{j,i+k-j} + (r^0)^{i,k}_{i,k}, j < i$, since $\mathrm{sign}(\alpha,\beta) = 1$ for all $\alpha \prec \beta$. Thus, it suffices to show $\mathrm{sign}(\alpha,\beta)\epsilon_{\alpha,\beta} + r(\alpha,\beta) = -\frac{2O(\alpha,\beta)}{n}$ where, as before, $O(\alpha,\beta) = m$ when $\tau^m\alpha = \beta$.

One sees that

$$\begin{aligned}(r^0)^{j,i+k-j}_{j,i+k-j} + (r^0)^{i,k}_{i,k} &= 1 - \frac{1}{2}[2j = i+k] - \frac{1}{2}[i = k] - \frac{1}{n}\mathrm{Res}(\frac{i+k-2j}{m}) - \frac{1}{n}\mathrm{Res}(\frac{k-i}{m}) \\ &= 1 - \frac{1}{2}[2j = i+k] - \frac{1}{2}[i = k] - \frac{2}{n}\mathrm{Res}(\frac{k-j}{m}) + M_{i,j,k}, \end{aligned} \tag{B.2}$$

where

$$M_{i,j,k} = [2j \neq i+k][\mathrm{Res}(\frac{k-j}{m}) > \mathrm{Res}(\frac{i+k-2j}{m})] - [i \neq k][\mathrm{Res}(\frac{k-j}{m}) < \mathrm{Res}(\frac{k-i}{m})]. \tag{B.3}$$

Thus, since $\mathrm{Res}(\frac{k-j}{m}) = O(\alpha,\beta)$, it suffices to show $1 + M_{i,j,k} + \epsilon_{\alpha,\beta} = 0$. In the case $\alpha \lessdot \beta$ or $\beta \lessdot \alpha$, it is easy to see that $\epsilon_{\alpha,\beta} = M_{i,j,k} = -\frac{1}{2}$. Otherwise, we may write

$$1 + M_{i,j,k} = [\mathrm{Res}(\frac{k-j}{m}) > \mathrm{Res}(\frac{i+k-2j}{m})] + [\mathrm{Res}(\frac{k-j}{m}) > \mathrm{Res}(\frac{k-i}{m})], \tag{B.4}$$

and it is easy to see that $[\mathrm{Res}(\frac{k-j}{m}) > \mathrm{Res}(\frac{i+k-2j}{m})] = [\exists\gamma, \alpha \prec \gamma \prec \beta, \alpha \gtrdot \gamma]$ while $[\mathrm{Res}(\frac{k-j}{m}) > \mathrm{Res}(\frac{k-i}{m})] = [\exists\gamma, \alpha \prec \gamma \prec \beta, \alpha \lessdot \gamma]$. This finishes the proof. $\square$

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